Random Effects (I)

Jarrod Hadfield

University of Edinburgh

Linear Model

- Model Syntax for Fixed Effects

```
y ~ limit + year + day
```

Linear Model

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```

- Set of Simultaneous Equations

$$
\begin{aligned}
E[y\texttt{[1]}] &= 1\beta_1 + (\texttt{limit[1]=="yes"})\beta_2 + (\texttt{year[1]=="1962"})\beta_3 + \texttt{day[1]}\beta_4 \\
E[y\texttt{[2]}] &= 1\beta_1 + (\texttt{limit[2]=="yes"})\beta_2 + (\texttt{year[2]=="1962"})\beta_3 + \texttt{day[2]}\beta_4 \\
\vdots &= \vdots \\
E[y\texttt{[184]}] &= 1\beta_1 + (\texttt{limit[184]=="yes"})\beta_2 + (\texttt{year[184]=="1962"})\beta_3 + \texttt{day[184]}\beta_4
\end{aligned}
$$

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E[y\texttt{[2]}] &= 1\beta_1 + (\texttt{limit[2]=="yes"})\beta_2 + (\texttt{year[2]=="1962"})\beta_3 + \texttt{day[2]}\,\beta_4 \\
\vdots &= \vdots \\
E[y\texttt{[184]}] &= 1\beta_1 + (\texttt{limit[184]=="yes"})\beta_2 + (\texttt{year[184]=="1962"})\beta_3 + \texttt{day[184]}\,\beta_4
\end{aligned}
$$

- Compact representation: design matrix and parameter vector

$$E[\mathbf{y}] = \mathbf{X}\beta$$

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\vdots &= \vdots \\
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\end{aligned}
$$

- Compact representation: design matrix and parameter vector

$$E[\mathbf{y}] = \mathbf{X}\boldsymbol{\beta}$$

```
> X <- model.matrix(y ~ limit + year + day, data = Traffic)
> X[c(1, 2, 184), ]
    (Intercept) limityes year1962 day
1             1        0        0   1
2             1        0        0   2
184           1        1        1  92
```

Linear Model

$$E[\mathbf{y}] = \mathbf{X}\beta$$

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- The full model

$$\mathbf{y} \sim N(\mathbf{X}\beta, \sigma^2_e\mathbf{I})$$

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- Residual structure

$$\sigma^2_e\mathbf{I} = \sigma^2_e \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sigma^2_e & 0 & \dots & 0 \\ 0 & \sigma^2_e & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sigma^2_e \end{bmatrix}$$

Linear Mixed Model

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y ~ as.factor(day)-1
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$$
\begin{aligned}
E[y\,[1]] &= \mathbf{X}[1,]\beta \\
E[y\,[2]] &= \mathbf{X}[2,]\beta \\
E[y\,[184]] &= \mathbf{X}[184,]\beta
\end{aligned}
$$

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$$
\begin{aligned}
E[y\texttt{[1]}] &= \mathbf{X}[1,]\beta + \texttt{(day[1]=="1")}u_1 + \texttt{(day[1]=="2")}u_2 \ldots \texttt{(day[1]=="92")}u_{92} \\
E[y\texttt{[2]}] &= \mathbf{X}[2,]\beta + \texttt{(day[2]=="1")}u_1 + \texttt{(day[2]=="2")}u_2 \ldots \texttt{(day[2]=="92")}u_{92} \\
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- Compact representation: design matrix and parameter vector

$$E[\mathbf{y}] = \mathbf{X}\beta + \mathbf{Z}\mathbf{u}$$

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$$E[\mathbf{y}] = \mathbf{X}\beta + \mathbf{Z}\mathbf{u}$$

$$\mathbf{W} = [\mathbf{X}, \mathbf{Z}]$$

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- Compact representation: design matrix and parameter vector

$$E[\mathbf{y}] = \mathbf{X}\beta + \mathbf{Z}\mathbf{u}$$

$$\mathbf{W} = [\mathbf{X}, \mathbf{Z}] \qquad \boldsymbol{\theta} = \begin{bmatrix} \beta \\ \mathbf{u} \end{bmatrix}$$

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$$E[\mathbf{y}] = \mathbf{W}\boldsymbol{\theta}$$

```
> Z <- model.matrix(~as.factor(day) - 1, data = Traffic)
> W <- cbind(X, Z)
```

Linear Mixed Model

- Fixed Effects

$$\beta \sim N(\mathbf{0}, \sigma^2_{\beta}\mathbf{I})$$

σ^2_{β} is not estimated, and is usually assumed to be large (or ∞ in non-Bayesian models)

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- Random Effects

$$\mathbf{u} \sim N(\mathbf{0}, \sigma^2_u \mathbf{I})$$

σ^2_u is estimated.

Linear Mixed Model

Fixed Effects

$\boldsymbol{\beta} \sim N(\mathbf{0}, \sigma^2_{\beta}\mathbf{I})$

σ^2_{β} not estimated (∞).

Random Effects

$\mathbf{u} \sim N(\mathbf{0}, \sigma^2_{u}\mathbf{I})$

σ^2_{u} estimated.

Linear Mixed Model

Fixed Effects	Random Effects
$\boldsymbol{\beta} \sim N(\mathbf{0}, \sigma^2_{\beta}\mathbf{I})$	$\mathbf{u} \sim N(\mathbf{0}, \sigma^2_{u}\mathbf{I})$
σ^2_{β} not estimated (∞).	σ^2_{u} estimated.

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The wrong sentence

- Nest was treated as random

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Linear Mixed Model

- Residual structure

$$\sigma^2_e\mathbf{I} = \sigma^2_e = \begin{bmatrix} \sigma^2_e & 0 & 0 & \dots & 0 \\ 0 & \sigma^2_e & 0 & \dots & 0 \\ 0 & 0 & \sigma^2_e & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \sigma^2_e \end{bmatrix}$$

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- Random-effect (G) structure

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- Random-effect (G) structure

$$\sigma^2_u\mathbf{Z}\mathbf{Z}^\top$$

Linear Mixed Model

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$$\sigma_e^2 \mathbf{I} = \sigma_e^2 = \begin{bmatrix} \sigma_e^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_e^2 & 0 & \dots & 0 \\ 0 & 0 & \sigma_e^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \sigma_e^2 \end{bmatrix}$$

- Random-effect (G) structure

$$\sigma_u^2 \mathbf{Z}\mathbf{Z}^\top = \sigma_u^2 \begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} =$$

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$$\sigma^2_u\mathbf{Z}\mathbf{Z}^\top = \sigma^2_u \begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sigma^2_u & \sigma^2_u & 0 & \dots & 0 \\ \sigma^2_u & \sigma^2_u & 0 & \dots & 0 \\ 0 & 0 & \sigma^2_u & \dots & \sigma^2_u \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \sigma^2_u & 0 & \sigma^2_u \end{bmatrix}$$

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Linear Mixed Model

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$$\sigma^2_u\mathbf{Z}\mathbf{Z}^\top = \begin{bmatrix} \sigma^2_u & \sigma^2_u & 0 & \dots & 0 \\ \sigma^2_u & \sigma^2_u & 0 & \dots & 0 \\ 0 & 0 & \sigma^2_u & \dots & \sigma^2_u \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \sigma^2_u & 0 & \sigma^2_u \end{bmatrix}$$

- Variance structure

$$\sigma^2_e\mathbf{I} + \sigma^2_u\mathbf{Z}\mathbf{Z}^\top = \begin{bmatrix} \sigma^2_e + \sigma^2_u & \sigma^2_u & 0 & \dots & 0 \\ \sigma^2_u & \sigma^2_e + \sigma^2_u & 0 & \dots & 0 \\ 0 & 0 & \sigma^2_e + \sigma^2_u & \dots & \sigma^2_u \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \sigma^2_u & 0 & \sigma^2_e + \sigma^2_u \end{bmatrix}$$

Linear Mixed Model

```
> traffic_m6 <- lmer(y ~ limit + year + day + (1 | day), data = Traffic)
```

Linear Mixed Model

```
> traffic_m6 <- lmer(y ~ limit + year + day + (1 | day), data = Traffic)
> summary(traffic_m6)
REML criterion at convergence: 1257.7

Scaled residuals:
     Min       1Q   Median       3Q      Max
-1.82638 -0.54453 -0.07602  0.59091  1.90812

Random effects:
 Groups   Name        Variance Std.Dev.
 day      (Intercept) 46.78    6.840
 Residual             25.74    5.074
Number of obs: 184, groups:  day, 92

Fixed effects:
            Estimate Std. Error t value
(Intercept) 21.59877    1.68542  12.815
limityes    -5.41959    0.95110  -5.698
year1962    -0.83338    0.79847  -1.044
day          0.05160    0.03033   1.701
```

Linear Mixed Model

```
> coef(summary(traffic_m6))
               Estimate Std. Error   t value
(Intercept) 21.59876943 1.68541510 12.815104
limityes    -5.41959096 0.95109962 -5.698237
year1962    -0.83338091 0.79847279 -1.043719
day          0.05159528 0.03033237  1.700998
```

Linear Mixed Model

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> coef(summary(traffic_m6))
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(Intercept) 21.59876943 1.68541510 12.815104
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day          0.05159528 0.03033237  1.700998
> coef(summary(traffic_m1))
               Estimate Std. Error   t value     Pr(>|t|)
(Intercept) 21.13110938 1.45169346 14.556179 3.131412e-32
limityes    -3.66426709 1.35558556 -2.703088 7.527902e-03
year1962    -1.34853031 1.31120928 -1.028463 3.051121e-01
day          0.05303589 0.02354966  2.252087 2.552498e-02
```
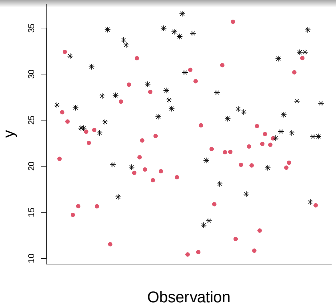
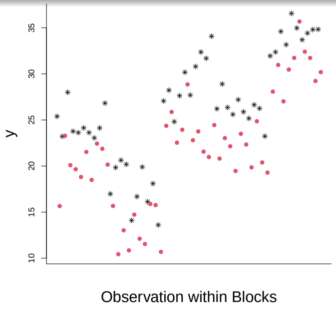

Linear Mixed Model: Fixed Effect Standard Errors



Linear Mixed Model: Fixed Effect Standard Errors



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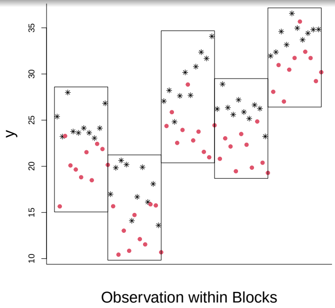

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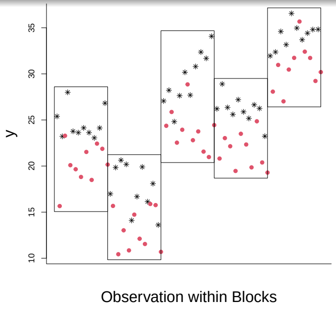
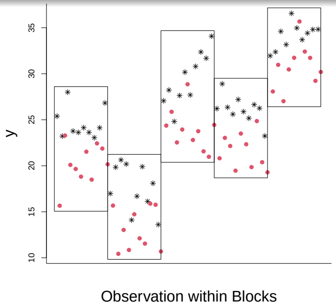
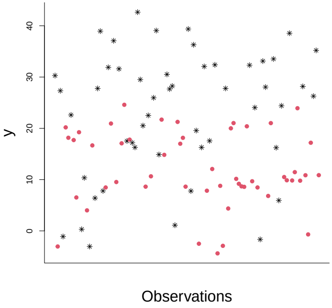


- Applying *both* treatments within blocks, standard error of the difference goes down.

Linear Mixed Model: Fixed Effect Standard Errors





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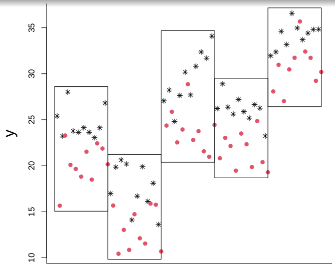
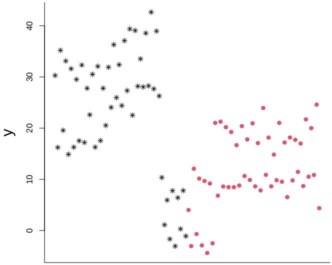


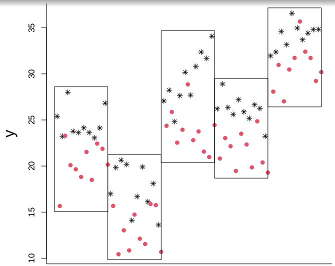
Observation within Blocks



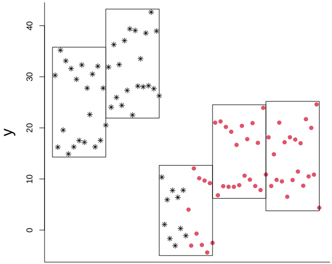
Observations within Days

- Applying *both* treatments within blocks, standard error of the difference goes down.

Linear Mixed Model: Fixed Effect Standard Errors



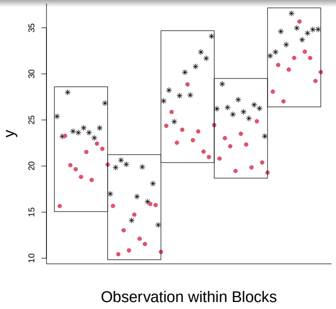
Observation within Blocks



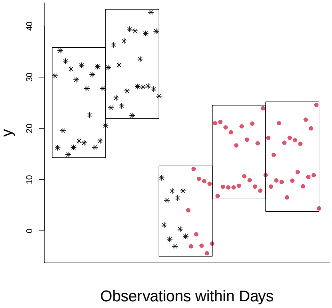
Observations within Days

- Applying *both* treatments within blocks, standard error of the difference goes down.

Linear Mixed Model: Fixed Effect Standard Errors



- Applying *both* treatments within blocks, standard error of the difference goes down.



- Applying *one* treatment *within* blocks, standard error of the difference goes up.

Linear Mixed Model: Fixed Effect Standard Errors

```
> coef(summary(traffic_m6))
               Estimate Std. Error   t value
(Intercept) 21.59876943 1.68541510 12.815104
limityes    -5.41959096 0.95109962 -5.698237
year1962    -0.83338091 0.79847279 -1.043719
day          0.05159528 0.03033237  1.700998
> coef(summary(traffic_m1))
               Estimate Std. Error   t value     Pr(>|t|)
(Intercept) 21.13110938 1.45169346 14.556179 3.131412e-32
limityes    -3.66426709 1.35558556 -2.703088 7.527902e-03
year1962    -1.34853031 1.31120928 -1.028463 3.051121e-01
day          0.05303589 0.02354966  2.252087 2.552498e-02
```

Linear Mixed Model: Fixed Effect Standard Errors

```
> coef(summary(traffic_m6))
              Estimate Std. Error   t value
(Intercept) 21.59876943 1.68541510 12.815104
limityes    -5.41959096 0.95109962 -5.698237
year1962    -0.83338091 0.79847279 -1.043719
day          0.05159528 0.03033237  1.700998
> coef(summary(traffic_m1))
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(Intercept) 21.13110938 1.45169346 14.556179 3.131412e-32
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year1962    -1.34853031 1.31120928 -1.028463 3.051121e-01
day          0.05303589 0.02354966  2.252087 2.552498e-02
```

- days which have different treatments (e.g. in 1961 there is a speed limit but not in 1962) are over-represented, so SE on `limityes` goes down.

Linear Mixed Model: Fixed Effect Standard Errors

```
> coef(summary(traffic_m6))
               Estimate Std. Error   t value
(Intercept) 21.59876943 1.68541510 12.815104
limityes    -5.41959096 0.95109962 -5.698237
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year1962    -1.34853031 1.31120928 -1.028463 3.051121e-01
day          0.05303589 0.02354966  2.252087 2.552498e-02
```

- days which have different treatments (e.g. in 1961 there is a speed limit but not in 1962) are over-represented, so SE on `limityes` goes down.
- every day has both years represented, so SE on `year1962` goes down.

Linear Mixed Model: Fixed Effect Standard Errors

```
> coef(summary(traffic_m6))

             Estimate Std. Error   t value
(Intercept) 21.59876943 1.68541510 12.815104
limityes    -5.41959096 0.95109962 -5.698237
year1962    -0.83338091 0.79847279 -1.043719
day          0.05159528 0.03033237  1.700998

> coef(summary(traffic_m1))

             Estimate Std. Error   t value     Pr(>|t|)
(Intercept) 21.13110938 1.45169346 14.556179 3.131412e-32
limityes    -3.66426709 1.35558556 -2.703088 7.527902e-03
year1962    -1.34853031 1.31120928 -1.028463 3.051121e-01
day          0.05303589 0.02354966  2.252087 2.552498e-02
```

- days which have different treatments (e.g. in 1961 there is a speed limit but not in 1962) are over-represented, so SE on `limityes` goes down.
- every day has both years represented, so SE on `year1962` goes down.
- day as a continuous variable has no within day variance only between day variance (by definition!), so SE on `day` goes up.

Linear Mixed Model: Fixed Effect Hypothesis testing

```
> coef(summary(traffic_m6))
               Estimate Std. Error   t value
(Intercept) 21.59876943 1.68541510 12.815104
limityes    -5.41959096 0.95109962 -5.698237
year1962    -0.83338091 0.79847279 -1.043719
day          0.05159528 0.03033237  1.700998
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Linear Mixed Model: Fixed Effect Hypothesis testing

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> coef(summary(traffic_m6))
               Estimate Std. Error   t value
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limityes    -5.41959096 0.95109962 -5.698237
year1962    -0.83338091 0.79847279 -1.043719
day          0.05159528 0.03033237  1.700998
```

z-test

```
> 2 * (1 - pnorm(abs(coef(summary(traffic_m6))["day", "t value"])))
[1] 0.08894342
```

Linear Mixed Model: Fixed Effect Hypothesis testing

```
> coef(summary(traffic_m6))

               Estimate Std. Error   t value
(Intercept) 21.59876943 1.68541510 12.815104
limityes    -5.41959096 0.95109962 -5.698237
year1962    -0.83338091 0.79847279 -1.043719
day          0.05159528 0.03033237  1.700998
```

z-test

```
> 2 * (1 - pnorm(abs(coef(summary(traffic_m6))["day", "t value"])))
[1] 0.08894342
```

t-test approximation

```
> pbkrtest::KRmodcomp(traffic_m6, cbind(0, 0, 0, 1))
```

Linear Mixed Model: Fixed Effect Hypothesis testing

```
> coef(summary(traffic_m6))

               Estimate Std. Error   t value
(Intercept) 21.59876943 1.68541510 12.815104
limityes    -5.41959096 0.95109962 -5.698237
year1962    -0.83338091 0.79847279 -1.043719
day          0.05159528 0.03033237  1.700998
```

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> 2 * (1 - pnorm(abs(coef(summary(traffic_m6))["day", "t value"])))
[1] 0.08894342
```

t-test approximation

```
> pbkrtest::KRmodcomp(traffic_m6, cbind(0, 0, 0, 1))

        stat     ndf     ddf F.scaling p.value
Ftest  2.8934  1.0000 89.9121         1  0.0924
```

Linear Mixed Model: Fixed Effect Hypothesis testing

```
> coef(summary(traffic_m6))

               Estimate Std. Error   t value
(Intercept) 21.59876943 1.68541510 12.815104
limityes    -5.41959096 0.95109962 -5.698237
year1962    -0.83338091 0.79847279 -1.043719
day          0.05159528 0.03033237  1.700998
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> 2 * (1 - pnorm(abs(coef(summary(traffic_m6))["day", "t value"])))
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t-test approximation

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> pbkrtest::KRmodcomp(traffic_m6, cbind(0, 0, 0, 1))

        stat     ndf     ddf F.scaling p.value
Ftest  2.8934  1.0000 89.9121         1  0.0924
```

Likelihood ratio test

```
> traffic_m7 <- lmer(y ~ limit + year + (1 | day), data = Traffic)
> anova(traffic_m6, traffic_m7)
```

Linear Mixed Model: Fixed Effect Hypothesis testing

```
> coef(summary(traffic_m6))

                Estimate Std. Error   t value
(Intercept) 21.59876943 1.68541510 12.815104
limityes    -5.41959096 0.95109962 -5.698237
year1962    -0.83338091 0.79847279 -1.043719
day          0.05159528 0.03033237  1.700998
```

z-test

```
> 2 * (1 - pnorm(abs(coef(summary(traffic_m6))["day", "t value"])))
[1] 0.08894342
```

t-test approximation

```
> pbkrtest::KRmodcomp(traffic_m6, cbind(0, 0, 0, 1))

         stat     ndf     ddf F.scaling p.value
Ftest  2.8934  1.0000 89.9121         1  0.0924
```

Likelihood ratio test

```
> traffic_m7 <- lmer(y ~ limit + year + (1 | day), data = Traffic)
> anova(traffic_m6, traffic_m7)

           npar    AIC    BIC  logLik deviance  Chisq Df Pr(>Chisq)
traffic_m7    5 1269.8 1285.9 -629.90   1259.8
traffic_m6    6 1268.9 1288.2 -628.44   1256.9 2.9122  1    0.08791
```

Deal with non-independence properly

Deal with non-independence properly

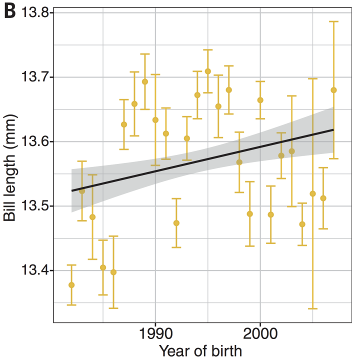
Science
AAAS

Recent natural selection causes adaptive evolution of an avian polygenic trait

Deal with non-independence properly

Science AAAS

Recent natural selection causes adaptive evolution of an avian polygenic trait



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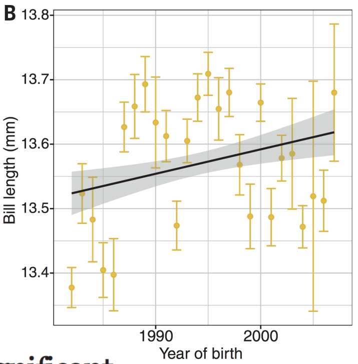
Science
AAAS

Recent natural selection causes adaptive evolution of an avian polygenic trait



we found that bill length has increased significantly over recent years (1982–2007; n = 2489 birds; estimate = 0.004 ± 0.001 mm per year; P = 0.0038;

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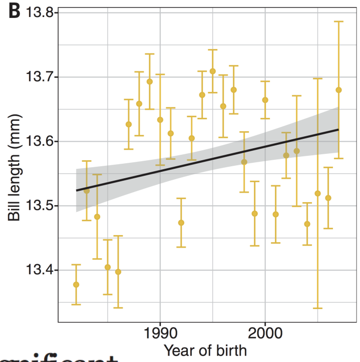
Recent natural selection causes adaptive evolution of an avian polygenic trait



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is not due to stochastic variation among years (randomization test, P = 0.02)

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Science AAAS

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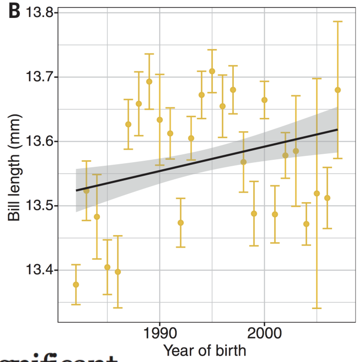

we found that bill length has increased significantly over recent years (1982–2007; $n = 2489$ birds; estimate = 0.004 ± 0.001 mm per year; $P = 0.0038$;

is not due to stochastic variation among years (randomization test, $P = 0.02$)

Bosse M *et. al* (2017) Data from: Recent natural selection causes adaptive evolution of an avian polygenic trait. Dryad Digital Repository. https://doi.org/10.5061/dryad.p03j0

Deal with non-independence properly

Science
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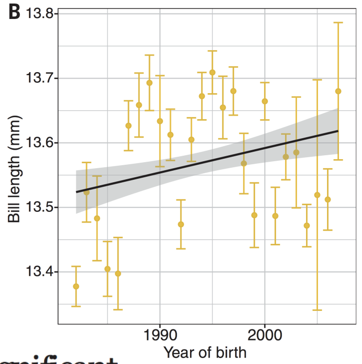
Recent natural selection causes adaptive evolution of an avian polygenic trait



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Bosse M *et. al* (2017) Data from: Recent natural selection causes adaptive evolution of an avian polygenic trait. Dryad Digital Repository. https://doi.org/10.5061/dryad.p03j0

Linear Mixed Model

```
> traffic_m6 <- lmer(y ~ limit + year + day + (1 | day), data = Traffic)
> summary(traffic_m6)
REML criterion at convergence: 1257.7

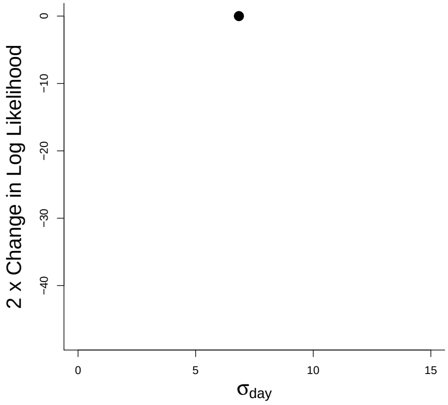
Scaled residuals:
     Min       1Q   Median       3Q      Max
-1.82638 -0.54453 -0.07602  0.59091  1.90812

Random effects:
 Groups   Name        Variance Std.Dev.
 day      (Intercept) 46.78    6.840
 Residual             25.74    5.074
Number of obs: 184, groups:  day, 92

Fixed effects:
            Estimate Std. Error t value
(Intercept) 21.59877    1.68542  12.815
limityes    -5.41959    0.95110  -5.698
year1962    -0.83338    0.79847  -1.044
day          0.05160    0.03033   1.701
```

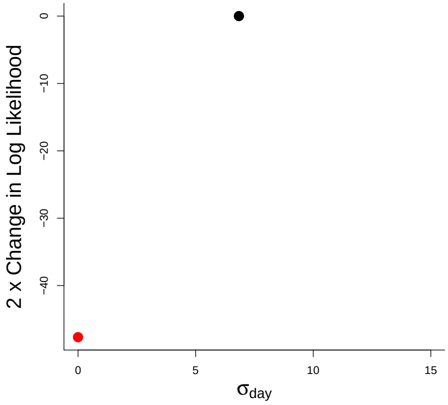
Linear Mixed Model: Confidence Intervals

Profile Likelihood



- Get the likelihood of the the model.

Profile Likelihood



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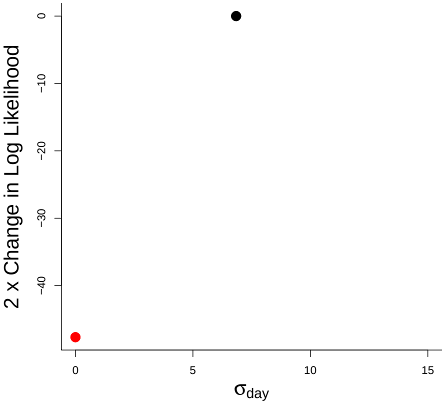
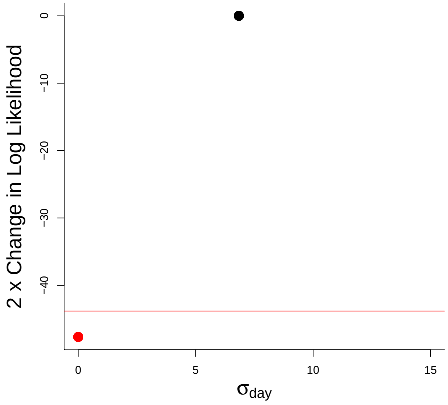


- Get the likelihood of the the model.
- Get the likelihood when σ^2_u is fixed at some value (and the other parameters, β and σ^2_e, are re-estimated).

Linear Mixed Model: Confidence Intervals

Profile Likelihood



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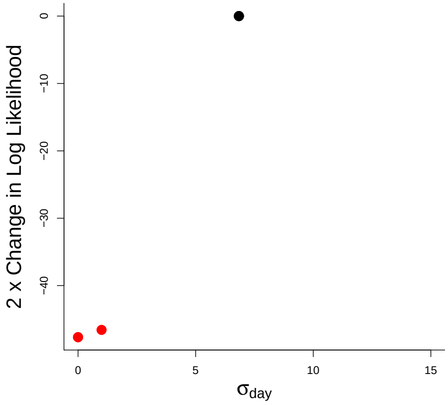

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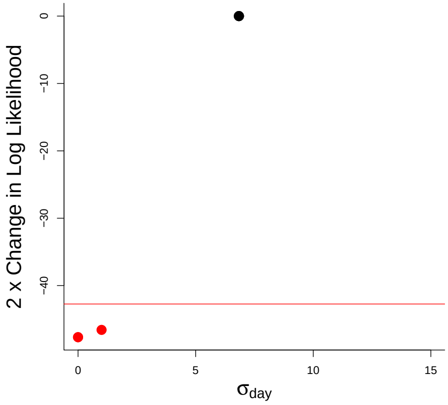
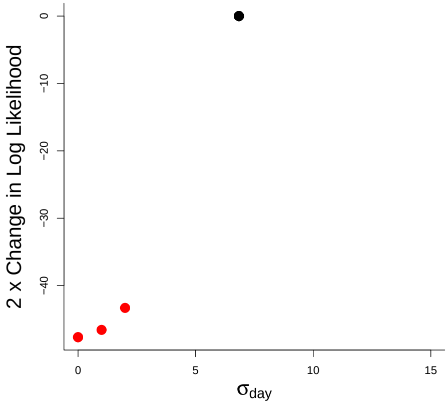
Linear Mixed Model: Confidence Intervals

Profile Likelihood



- Get the likelihood of the the model.
- Get the likelihood when σ_u^2 is fixed at some value (and the other parameters, β and σ_e^2, are re-estimated).

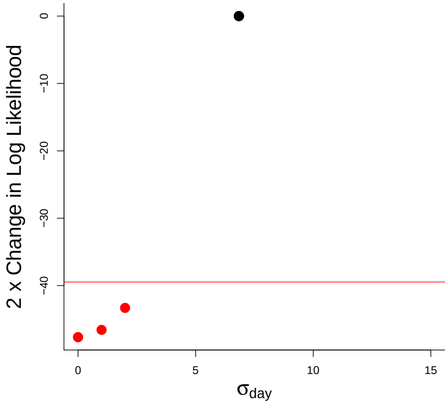
Profile Likelihood



- Get the likelihood of the the model.
- Get the likelihood when σ^2_u is fixed at some value (and the other parameters, β and σ^2_e, are re-estimated).

Linear Mixed Model: Confidence Intervals

Profile Likelihood



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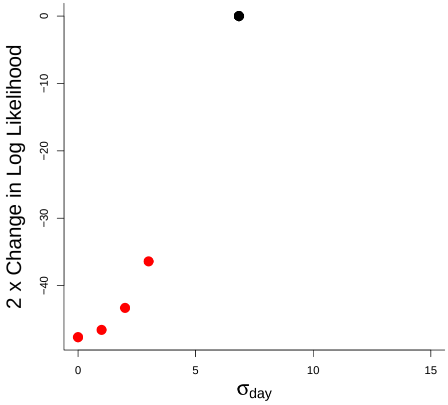

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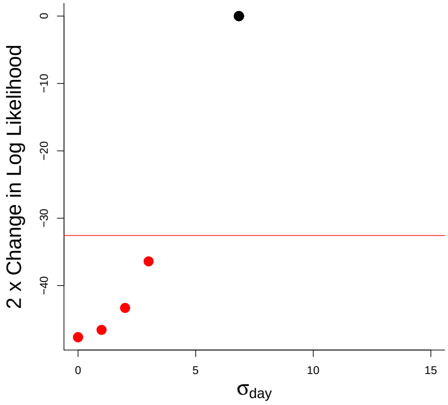

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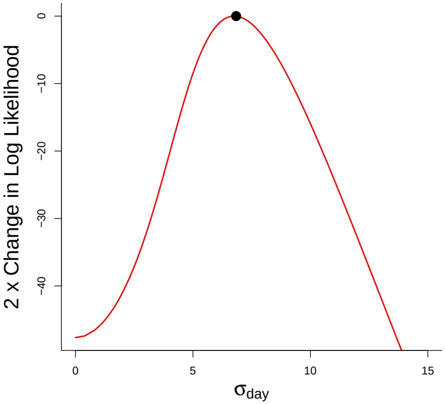

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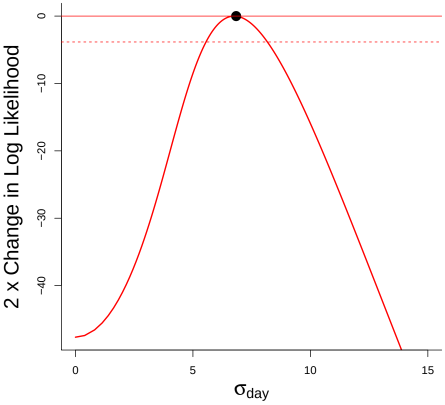

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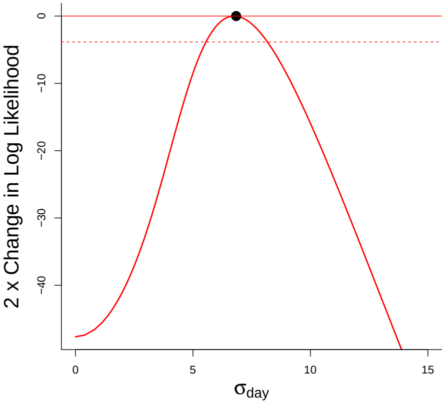
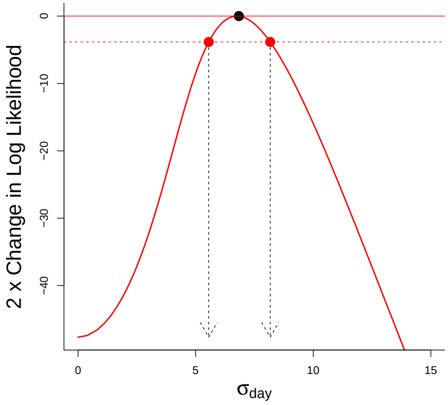
Profile Likelihood



- Get the likelihood of the the model.
- Get the likelihood when σ^2_u is fixed at some value (and the other parameters, β and σ^2_e, are re-estimated).
- The critical value is `qchisq(0.95,1)`=3.84 for the 95% confidence interval.

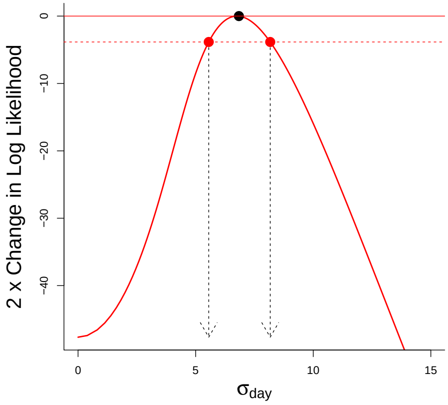
Linear Mixed Model: Confidence Intervals

Profile Likelihood



- Get the likelihood of the the model.
- Get the likelihood when σ^2_u is fixed at some value (and the other parameters, β and σ^2_e, are re-estimated).
- The critical value is `qchisq(0.95,1)`=3.84 for the 95% confidence interval.
- Find the values σ^2_u at which twice the difference in the log-likelihood is equal to some critical value.

Profile Likelihood




```
> confint(traffic_m6)[".sig01", ]
   2.5 %   97.5 %
5.556965 8.170629
```

- Get the likelihood of the the model.
- Get the likelihood when σ^2_u is fixed at some value (and the other parameters, β and σ^2_e, are re-estimated).
- The critical value is `qchisq(0.95,1)`=3.84 for the 95% confidence interval.
- Find the values σ^2_u at which twice the difference in the log-likelihood is equal to some critical value.

Linear Mixed Model: Confidence Intervals

Parametric Bootstrap

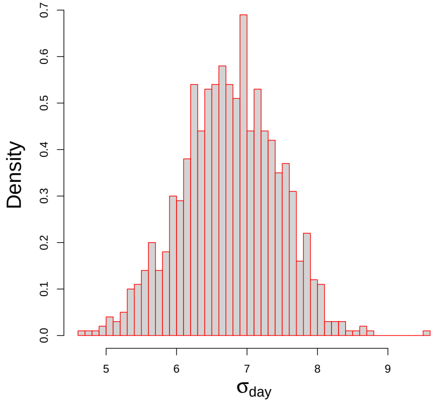
- Simulate new data using the (restricted) maximum likelihood estimates ($\hat{\beta}$, $\hat{\sigma}^2_e$ and $\hat{\sigma}^2_u$).

Linear Mixed Model: Confidence Intervals

Parametric Bootstrap

- Simulate new data using the (restricted) maximum likelihood estimates ($\hat{\beta}$, $\hat{\sigma}^2_e$ and $\hat{\sigma}^2_u$).
- Refit model using new data to get the sampling distribution.

Parametric Bootstrap



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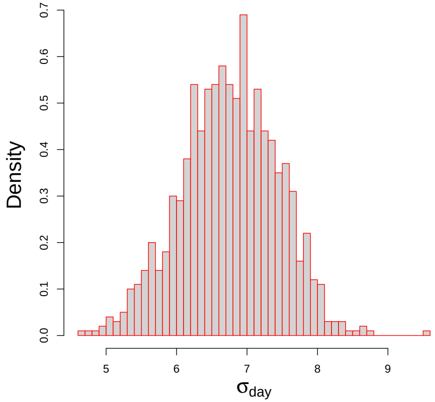
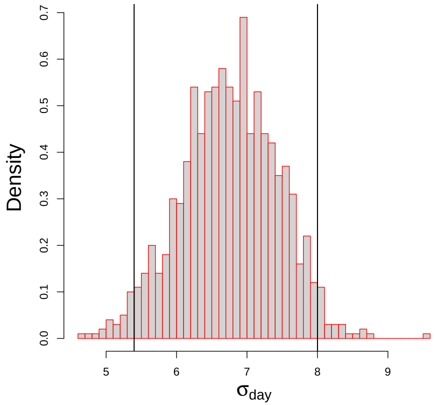
Parametric Bootstrap



- Simulate new data using the (restricted) maximum likelihood estimates ($\hat{\beta}$, $\hat{\sigma}^2_e$ and $\hat{\sigma}^2_u$).
- Refit model using new data to get the sampling distribution.
- Get the 2.5% and 97.5% quantiles

Parametric Bootstrap



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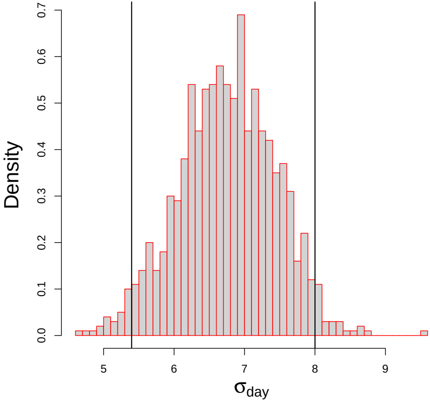
Parametric Bootstrap

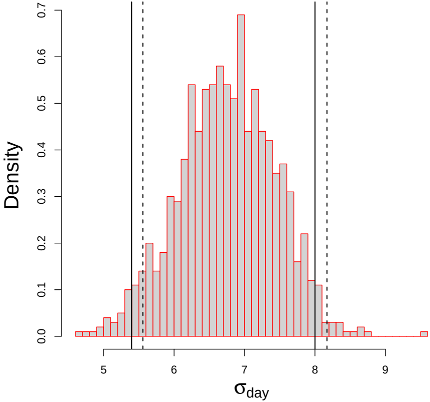


- Simulate new data using the (restricted) maximum likelihood estimates ($\hat{\beta}$, $\hat{\sigma}^2_e$ and $\hat{\sigma}^2_u$).
- Refit model using new data to get the sampling distribution.
- Get the 2.5% and 97.5% quantiles

```
> confint(traffic_m6, method = "boot")[".sig01", ]
   2.5 %   97.5 %
5.624805 8.010794
```

Linear Mixed Model: Confidence Intervals

Parametric Bootstrap



- Simulate new data using the (restricted) maximum likelihood estimates ($\hat{\beta}$, $\hat{\sigma}^2_e$ and $\hat{\sigma}^2_u$).
- Refit model using new data to get the sampling distribution.
- Get the 2.5% and 97.5% quantiles

```
> confint(traffic_m6, method = "boot")[".sig01", ]
   2.5 %   97.5 %
5.624805 8.010794
```

Parametric Bootstrap

```
> traffic_sim <- simulate(traffic_m6, 1000)
```

Linear Mixed Model: Confidence Intervals

Parametric Bootstrap

```
> traffic_sim <- simulate(traffic_m6, 1000)
> traffic_pb <- t(apply(traffic_sim, 2, function(x) {
+     coefv(refit(traffic_m6, x))
+ }))
```

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> head(traffic_pb)

      day.(Intercept) Residual
sim_1        38.69853 30.42440
sim_2        43.12451 26.22378
sim_3        56.11658 22.15043
sim_4        52.20232 26.28133
sim_5        44.87838 23.92620
sim_6        44.30703 25.31061
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> propday.pb <- traffic_pb[, 1]/rowSums(traffic_pb)
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sim_6        44.30703 25.31061
> propday.pb <- traffic_pb[, 1]/rowSums(traffic_pb)
> quantile(propday.pb, prob = c(0.025, 0.975))
     2.5%     97.5%
0.4970392 0.7504978
```

Likelihood Ratio Test

Linear Mixed Model: Random effect hypothesis testing

Likelihood Ratio Test

```
> anova(traffic_m6, traffic_m1)

refitting model(s) with ML (instead of REML)

           npar    AIC    BIC  logLik deviance  Chisq Df Pr(>Chisq)
traffic_m1    5 1314.5 1330.6 -652.27   1304.5
traffic_m6    6 1268.9 1288.2 -628.44   1256.9 47.656  1  5.081e-12
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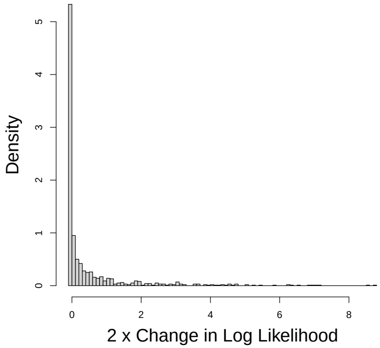

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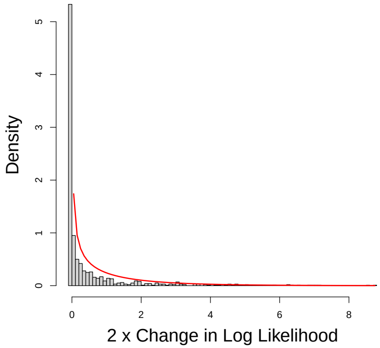

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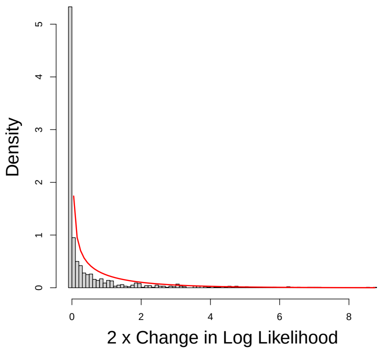

- 50% chance of being zero (observations from the same day are dissimilar).

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Likelihood Ratio Test

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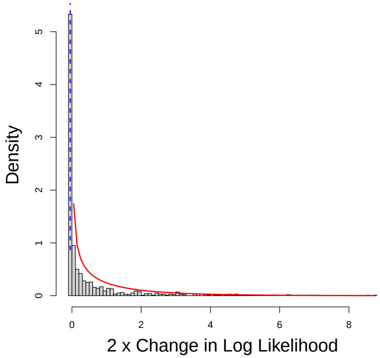

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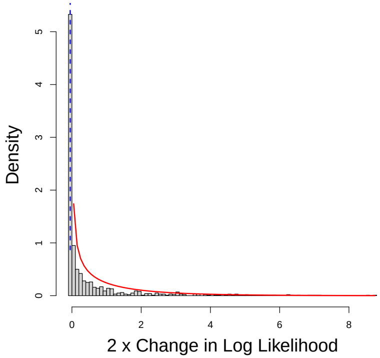

- 50% chance of being zero (observations from the same day are dissimilar).
- 50% chance of being non-zero and following a chi-squared distribution with 1 degree of freedom.

Linear Mixed Model: Random effect hypothesis testing

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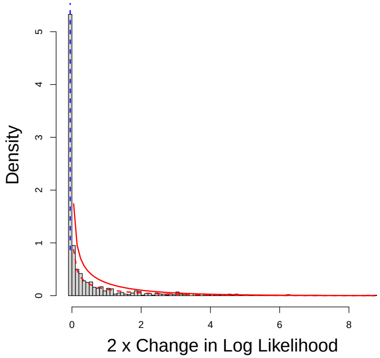

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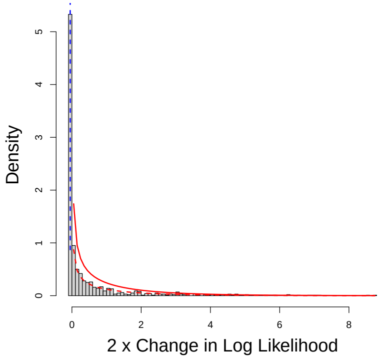

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- Halve the p-value

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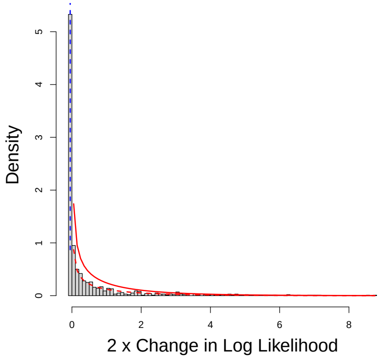
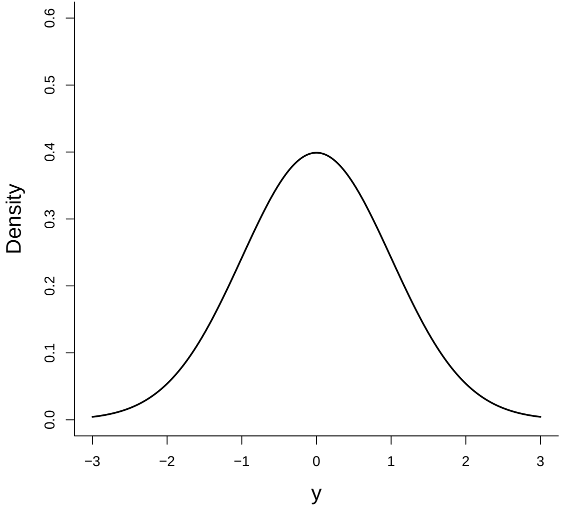


- 50% chance of being zero (observations from the same day are dissimilar).
- 50% chance of being non-zero and following a chi-squared distribution with 1 degree of freedom.
- Halve the p-value
- Don't do this if you are fitting covariance matrices.

ML versus REML

ML versus REML



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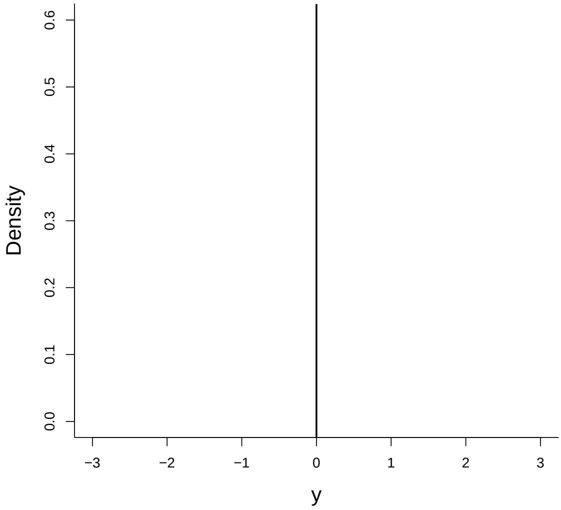

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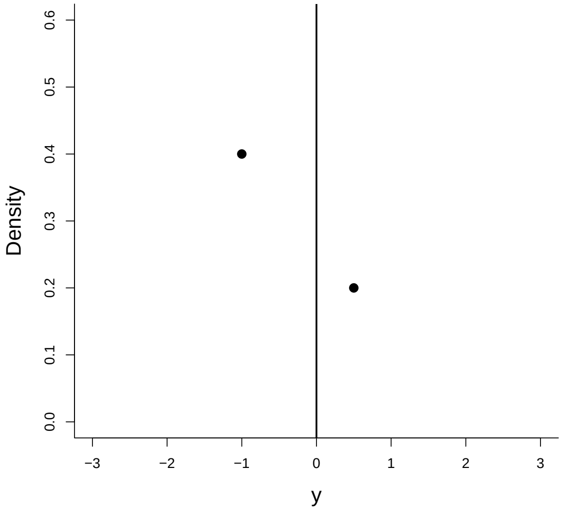

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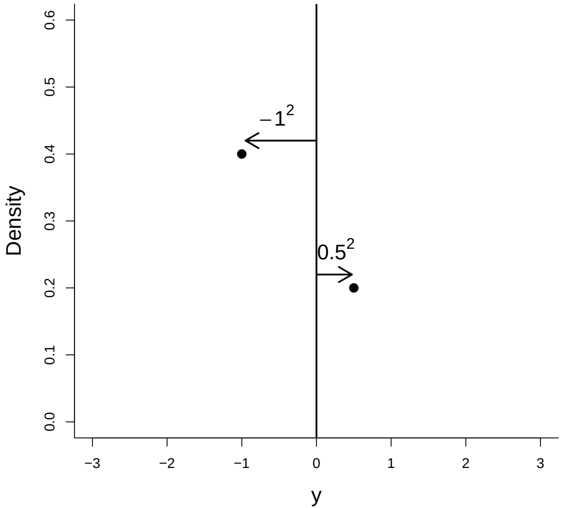

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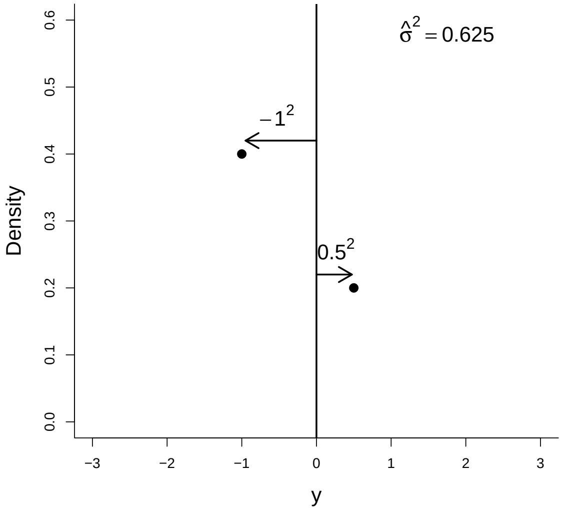

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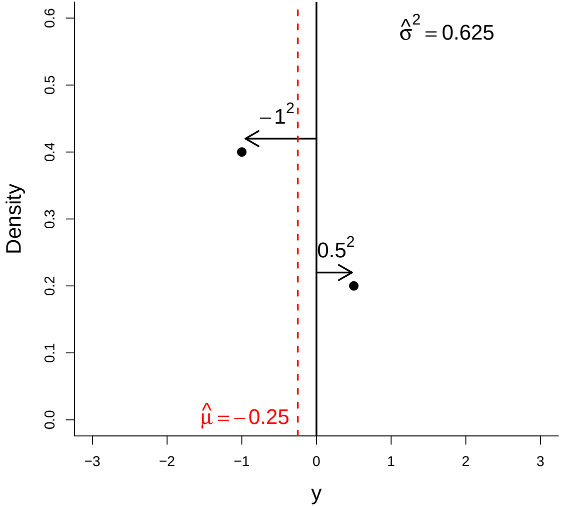

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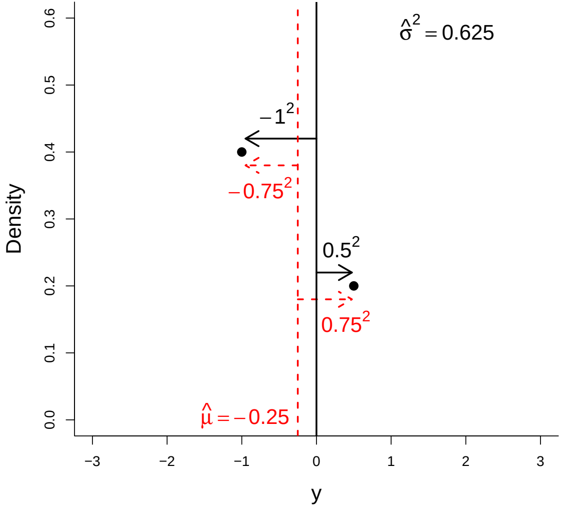

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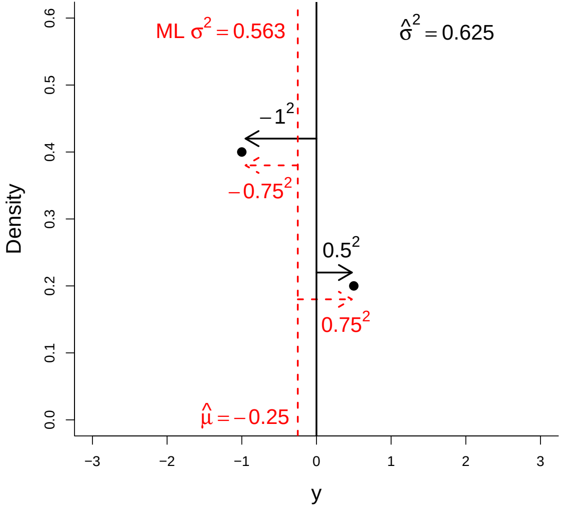

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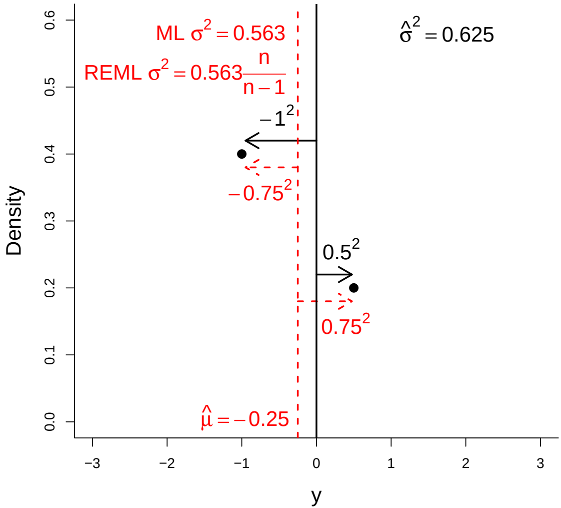

Linear Mixed Model: Likelihood-based tests

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- Models compared via likelihood ratio tests must be *nested* (i.e the simpler model must be a special case of the more complex model)

Many functions, such as `anova` now check but if you are doing things 'by hand' you need to be careful.

Are they fixed or random?

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```
> head(BTtarsus, 2)

  tarsus_mm bird_id sex year nest_orig nest_rear day_hatch
1      17.2 L298904   F 2011     11_A9     11_A9         0
2      17.6 L298903   M 2011     11_A9     11_A9         0
```

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> tarsus_m1 <- lm(tarsus_mm ~ sex + day_hatch +
+     year, data = BTtarsus)
```

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> anova(tarsus_m1, tarsus_null)

  Res.Df    RSS Df Sum of Sq      F    Pr(>F)
1   2902 917.14
2   2905 923.49 -3   -6.3475 6.6949 0.0001682
```

Are they fixed or random?

```
> head(BTtarsus, 2)
  tarsus_mm bird_id sex year nest_orig nest_rear day_hatch
1      17.2 L298904   F 2011      11_A9     11_A9         0
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> tarsus_m2 <- lmer(tarsus_mm ~ sex + day_hatch +
+     (1 | year), data = BTtarsus, REML = FALSE)
> tarsus_null <- update(tarsus_m1, . ~ . - year)
```

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> anova(tarsus_m2, tarsus_null)
            npar    AIC    BIC  logLik deviance  Chisq Df
tarsus_null    4 4924.9 4948.8 -2458.4   4916.9
tarsus_m2      5 4917.6 4947.5 -2453.8   4907.6 9.2717  1
            Pr(>Chisq)
tarsus_null
tarsus_m2     0.002327
```

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tarsus_null
tarsus_m2     0.002327
> RLRsim::exactLRT(tarsus_m2, tarsus_null)
LRT = 9.2717, p-value = 0.00017
```

Are they fixed or random?

```
> fixef(tarsus_m2)[1]
(Intercept)
   16.65926
> ranef(tarsus_m2)[1]
$year
       (Intercept)
2011  0.0319286735
2012  0.0002950763
2013 -0.0642531743
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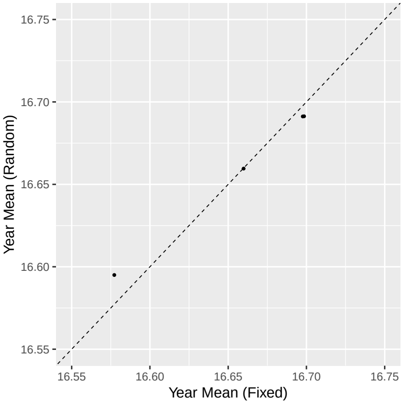


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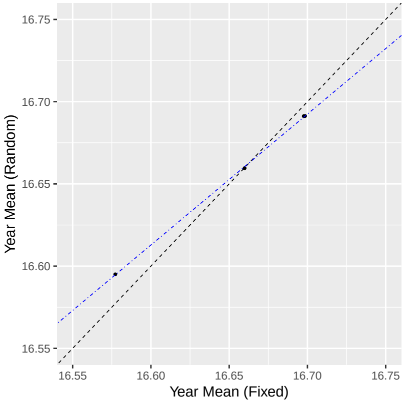


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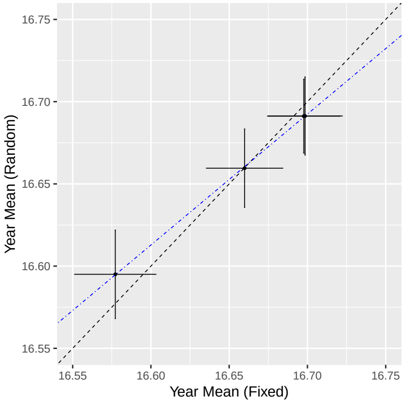


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