Random Effects (II)

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Multiple Random Effects?

```
> head(BTtarsus)

  tarsus_mm bird_id sex year nest_orig nest_rear day_hatch
1      17.2 L298904   F 2011     11_A9     11_A9         0
2      17.6 L298903   M 2011     11_A9     11_A9         0
3      16.2 L298905   F 2011     11_82     11_A9         0
4      17.0 L298901   M 2011     11_82     11_A9         0
5      17.3 L298900   M 2011     11_A9     11_A9         1
6      16.1 L298902   M 2011     11_82     11_A9         1
```

Multiple Random Effects: Nested & Cross-classified

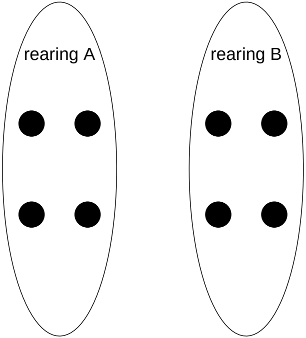
```
> tarsus_m5 <- lmer(tarsus_mm ~ sex + day_hatch +
+     year + (1 | nest_orig) + (1 | nest_rear),
+     data = BTtarsus)
```

Multiple Random Effects: Nested & Cross-classified

```
> tarsus_m5 <- lmer(tarsus_mm ~ sex + day_hatch +
+      year + (1 | nest_orig) + (1 | nest_rear),
+      data = BTtarsus)

> summary(tarsus_m5)

REML criterion at convergence: 3493.7

Scaled residuals:
    Min      1Q  Median      3Q     Max
-5.2498 -0.5696  0.0162  0.6117  3.2833

Random effects:
 Groups    Name        Variance Std.Dev.
 nest_orig (Intercept) 0.07971  0.2823
 nest_rear (Intercept) 0.13642  0.3693
 Residual              0.12963  0.3600
Number of obs: 2908, groups:
nest_orig, 440; nest_rear, 358

Fixed effects:
```

Multiple Random Effects: Nested & Cross-classified

Cross-classified



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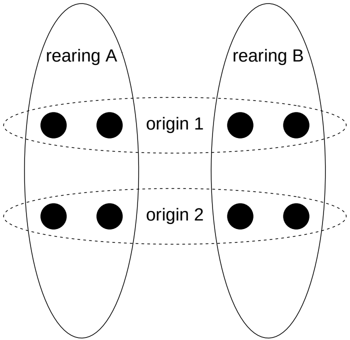

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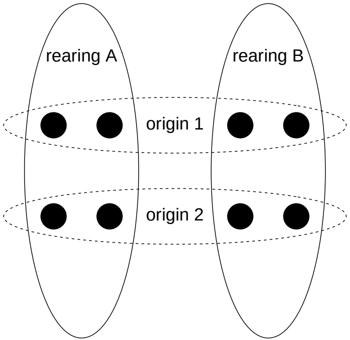
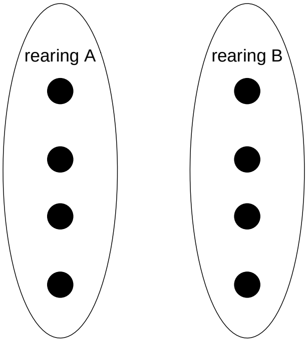


Nested



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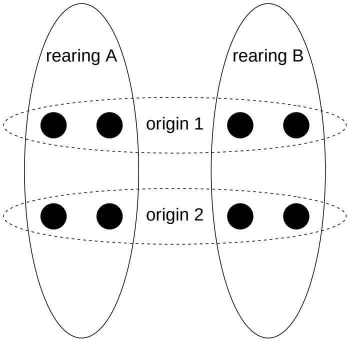
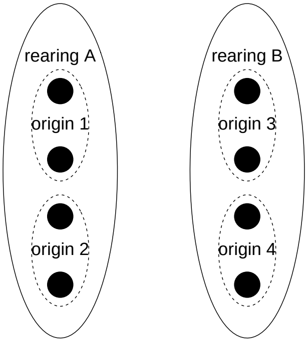
Cross-classified



Nested



Multiple Random Effects: Nested & Cross-classified

y	Nest	ID
1	A_11	1
2	A_11	1
5	A_11	2
6	A_11	2
4	A_11	3
3	A_11	3
1	A_08	1
1	A_08	1
2	A_08	2
4	A_08	2
6	A_16	1
⋮	⋮	⋮

Multiple Random Effects: Nested & Cross-classified

y	Nest	ID
1	A_11	1
2	A_11	1
5	A_11	2
6	A_11	2
4	A_11	3
3	A_11	3
1	A_08	1
1	A_08	1
2	A_08	2
4	A_08	2
6	A_16	1
⋮	⋮	⋮

```
y ~ (1|Nest)+(1|ID)
```

Multiple Random Effects: Nested & Cross-classified

y	Nest	ID
1	A_11	1
2	A_11	1
5	A_11	2
6	A_11	2
4	A_11	3
3	A_11	3
1	A_08	1
1	A_08	1
2	A_08	2
4	A_08	2
6	A_16	1
⋮	⋮	⋮

```
y ~ (1|Nest)+(1|ID)
y ~ (1|Nest)+(1|Nest:ID)
```

Multiple Random Effects: Nested & Cross-classified

y	Nest	ID
1	A_11	1
2	A_11	1
5	A_11	2
6	A_11	2
4	A_11	3
3	A_11	3
1	A_08	1
1	A_08	1
2	A_08	2
4	A_08	2
6	A_16	1
⋮	⋮	⋮

```
y ~ (1|Nest)+(1|ID)
y ~ (1|Nest)+(1|Nest:ID)
y ~ (1|Nest)+(1|Nest/ID)
```

Multiple Random Effects: Nested & Cross-classified

y	Nest	ID
1	A_11	1
2	A_11	1
5	A_11	2
6	A_11	2
4	A_11	3
3	A_11	3
1	A_08	1
1	A_08	1
2	A_08	2
4	A_08	2
6	A_16	1
⋮	⋮	⋮

y	Nest	ID
1	A_11	A_11_1
2	A_11	A_11_1
5	A_11	A_11_2
6	A_11	A_11_2
4	A_11	A_11_3
3	A_11	A_11_3
1	A_08	A_08_1
1	A_08	A_08_1
2	A_08	A_08_2
4	A_08	A_08_2
6	A_16	A_16_1
⋮	⋮	⋮

```
y ~ (1|Nest)+(1|ID)
y ~ (1|Nest)+(1|Nest:ID)
y ~ (1|Nest)+(1|Nest/ID)
```

Multiple Random Effects: Nested & Cross-classified

y	Nest	ID
1	A_11	1
2	A_11	1
5	A_11	2
6	A_11	2
4	A_11	3
3	A_11	3
1	A_08	1
1	A_08	1
2	A_08	2
4	A_08	2
6	A_16	1
⋮	⋮	⋮

```
y ~ (1|Nest)+(1|ID)
y ~ (1|Nest)+(1|Nest:ID)
y ~ (1|Nest)+(1|Nest/ID)
```

y	Nest	ID
1	A_11	A_11_1
2	A_11	A_11_1
5	A_11	A_11_2
6	A_11	A_11_2
4	A_11	A_11_3
3	A_11	A_11_3
1	A_08	A_08_1
1	A_08	A_08_1
2	A_08	A_08_2
4	A_08	A_08_2
6	A_16	A_16_1
⋮	⋮	⋮

```
y ~ (1|Nest)+(1|ID)
```

Generalised Linear Mixed Model

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$$E[\mathbf{y}] = \mathbf{X}\boldsymbol{\beta}$$

Generalised Linear Mixed Model

- Link function: log

$$E[\mathbf{y}] = \exp(\mathbf{X}\boldsymbol{\beta})$$

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- Distribution: Poisson

$$\mathbf{y} \sim Pois(\exp(\mathbf{X}\beta))$$

Generalised Linear Mixed Model

- Link function: log

$$E[\mathbf{y}] = \exp(\mathbf{W}\boldsymbol{\theta})$$

- Distribution: Poisson

$$\mathbf{y} \sim Pois(\exp(\mathbf{W}\boldsymbol{\theta}))$$

Generalised Linear Model: Overdispersion

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```
> Traffic$obs <- as.factor(1:nrow(Traffic))
```

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```
> Traffic$obs <- as.factor(1:nrow(Traffic))

> traffic_m8 <- glmer(y ~ limit + year + day + (1 | obs), data = Traffic,
+     family = poisson)
```

Generalised Linear Model: Overdispersion

```
> Traffic$obs <- as.factor(1:nrow(Traffic))

> traffic_m8 <- glmer(y ~ limit + year + day + (1 | obs), data = Traffic,
+     family = poisson)

> summary(traffic_m8)
     AIC      BIC   logLik deviance df.resid
  1284.2   1300.3   -637.1   1274.2      179

Scaled residuals:
     Min       1Q   Median       3Q      Max
-1.61630 -0.43342 -0.07274  0.36395  1.15236

Random effects:
 Groups Name        Variance Std.Dev.
 obs    (Intercept) 0.09613  0.31
Number of obs: 184, groups:  obs, 184

Fixed effects:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)  2.991249   0.065909   45.38  < 2e-16 ***
limityes    -0.170256   0.061477   -2.77  0.00562 **
year1962    -0.065551   0.058947   -1.11  0.26612
day          0.002580   0.001067    2.42  0.01558 *
---
```

Generalised Linear Model: Overdispersion

- Random-effect (G) structure

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$\sigma^2_u \mathbf{Z}\mathbf{Z}^\top$

Generalised Linear Model: Overdispersion

- Random-effect (G) structure

$$\sigma^2_u \mathbf{Z}\mathbf{Z}^\top = \sigma^2_u \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Generalised Linear Model: Overdispersion

- Random-effect (G) structure

$$\sigma^2_u\mathbf{Z}\mathbf{Z}^\top = \sigma^2_u\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sigma^2_u & 0 & 0 & \dots & 0 \\ 0 & \sigma^2_u & 0 & \dots & 0 \\ 0 & 0 & \sigma^2_u & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \sigma^2_u \end{bmatrix}$$

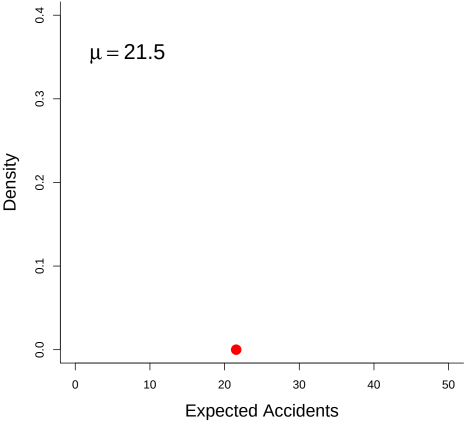
Generalised Linear Model: Overdispersion

- Random-effect (G) structure

$$\sigma^2_u\mathbf{Z}\mathbf{Z}^\top = \sigma^2_u\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sigma^2_u & 0 & 0 & \dots & 0 \\ 0 & \sigma^2_u & 0 & \dots & 0 \\ 0 & 0 & \sigma^2_u & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \sigma^2_u \end{bmatrix}$$

Like fitting a residual

Understanding the Negative Binomial



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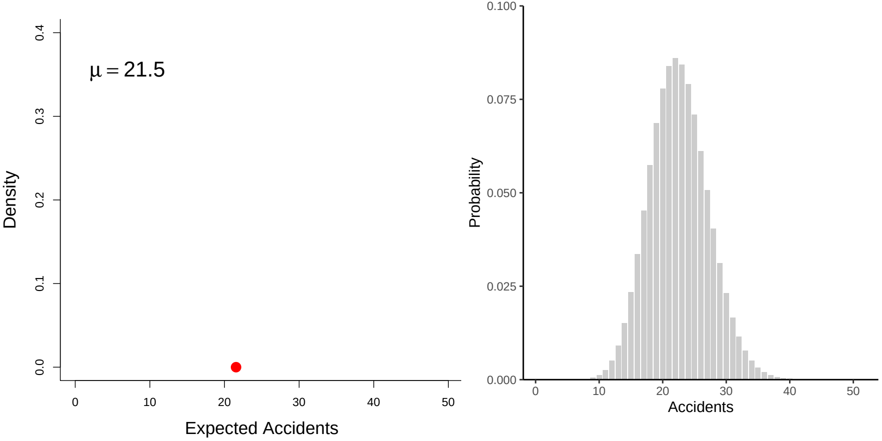

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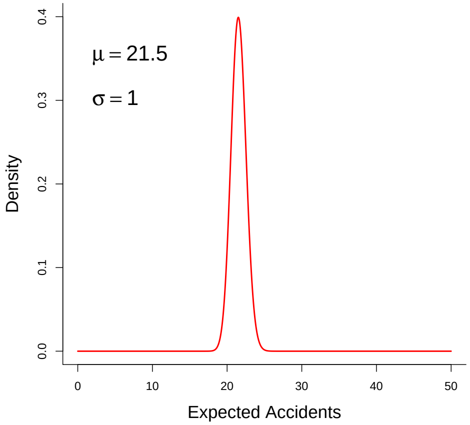

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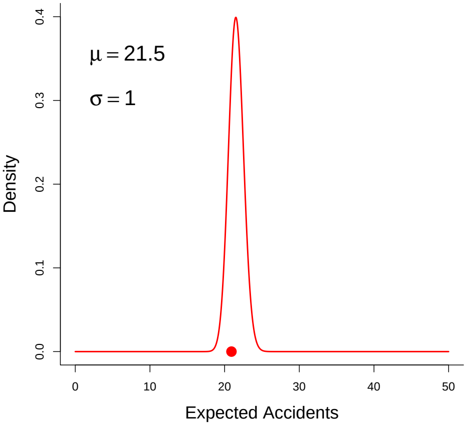

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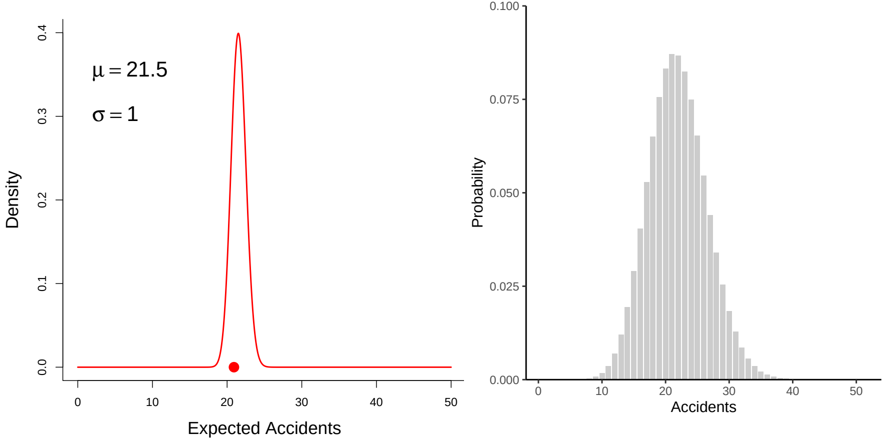

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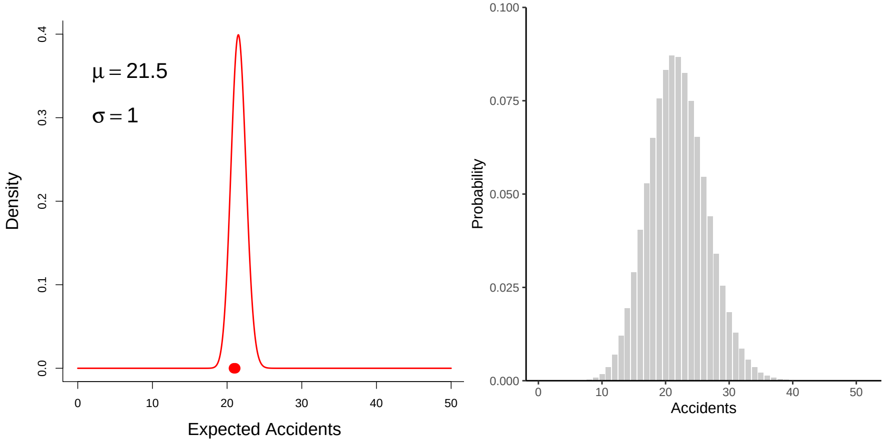

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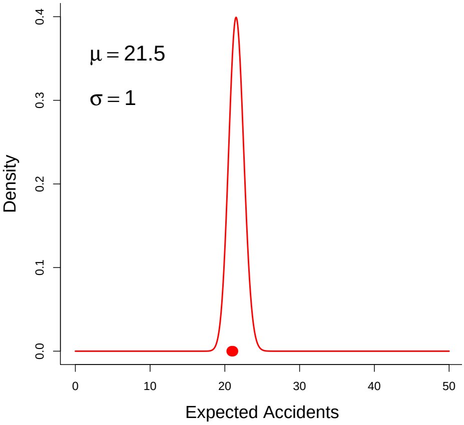

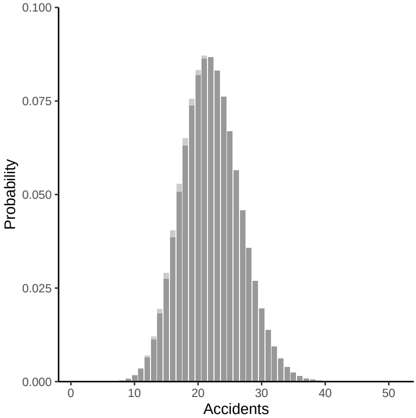

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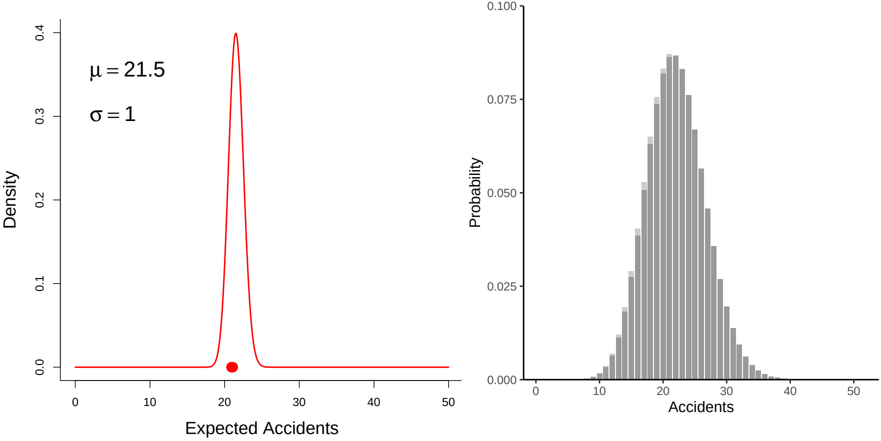

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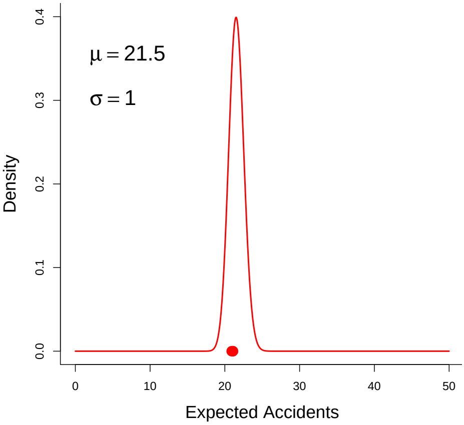

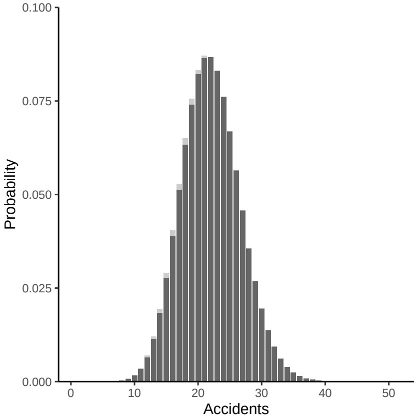

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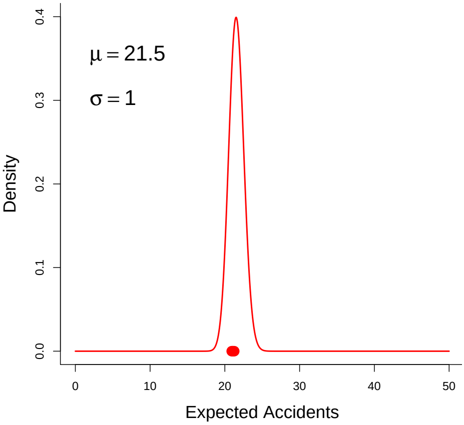

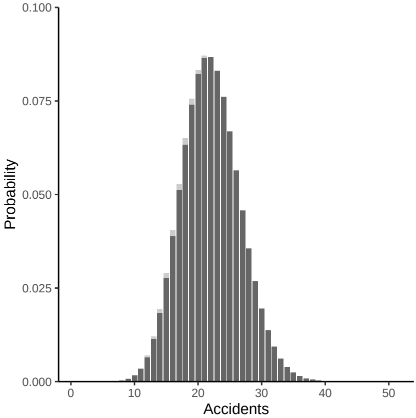

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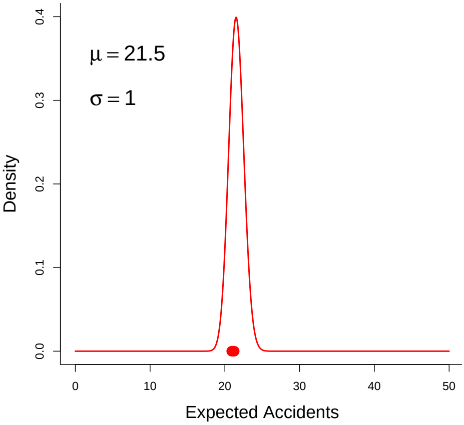

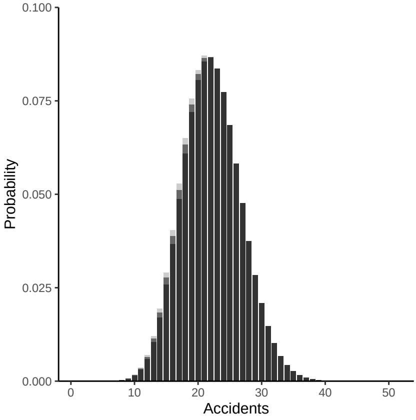

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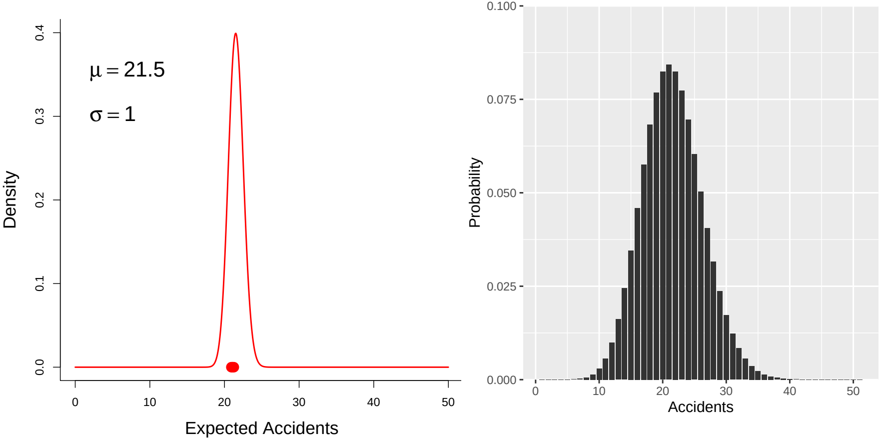

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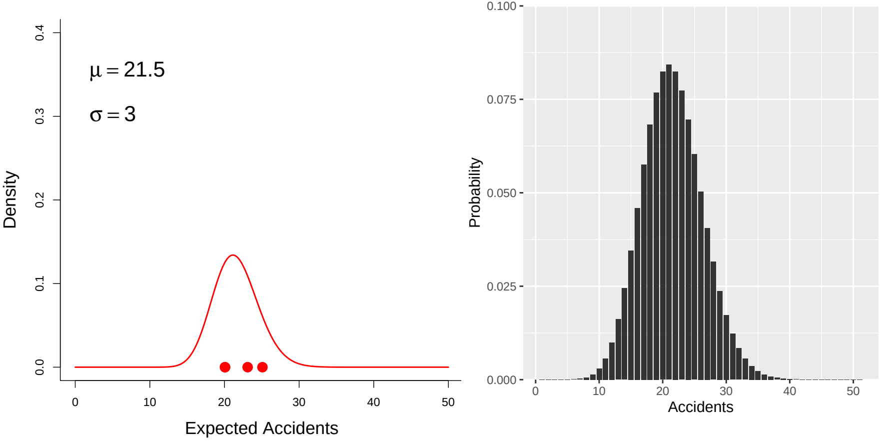

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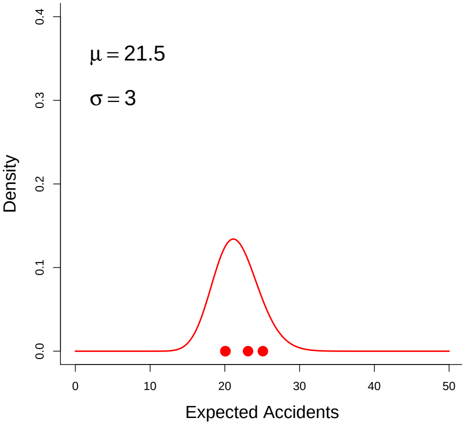

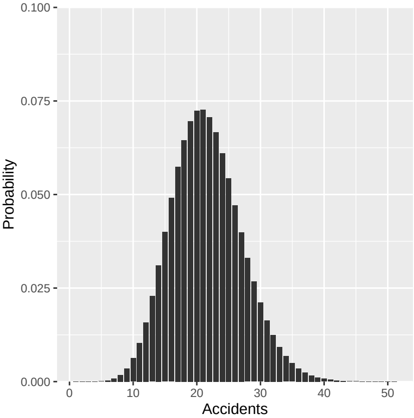

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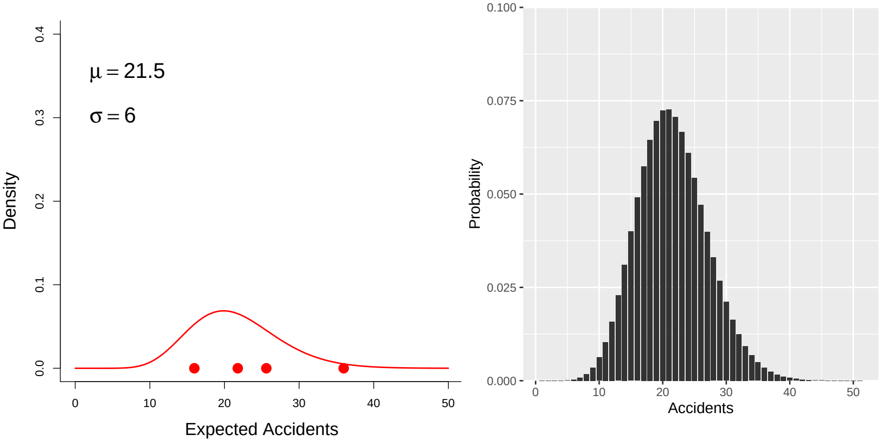

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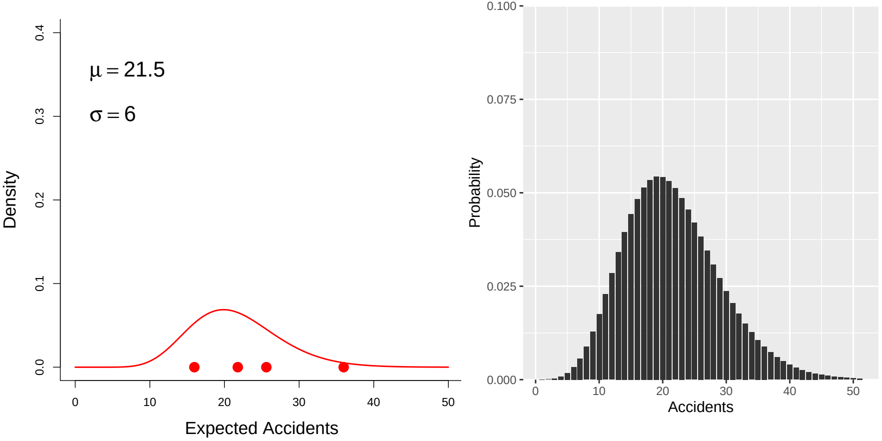

Understanding the GLMM

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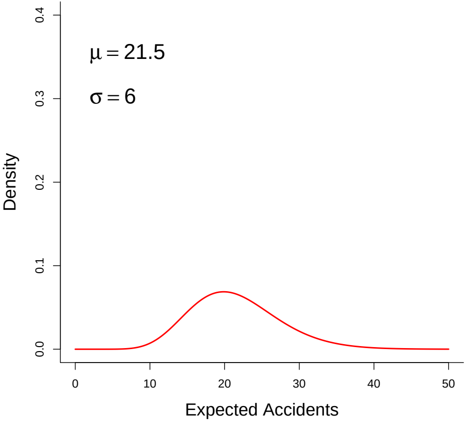

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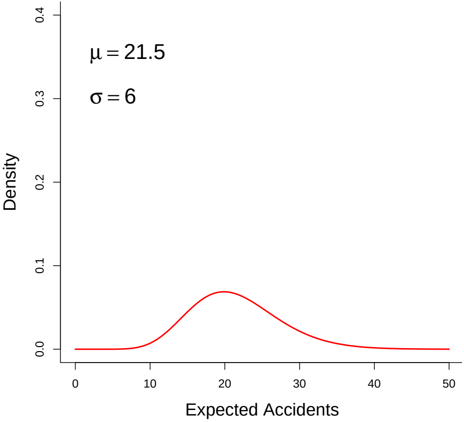

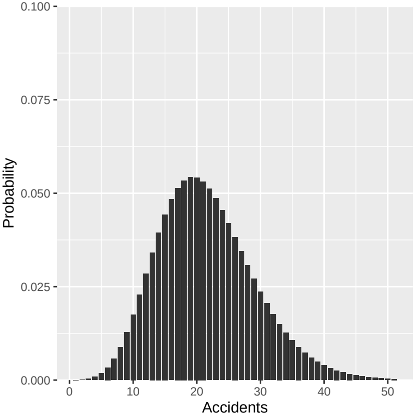

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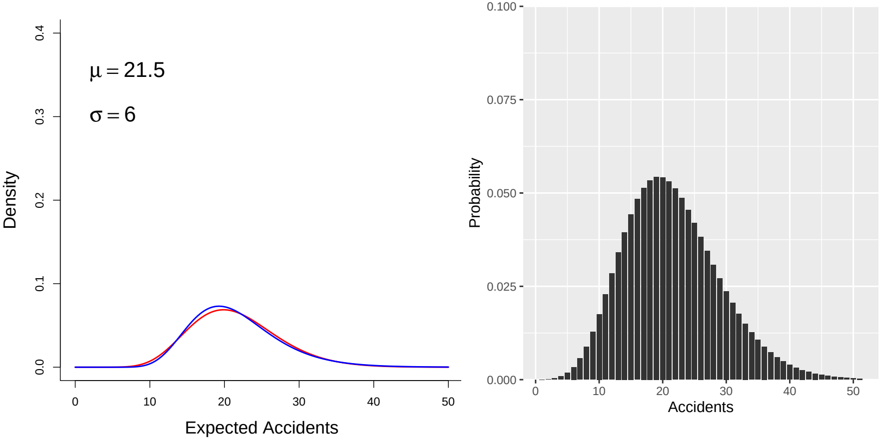

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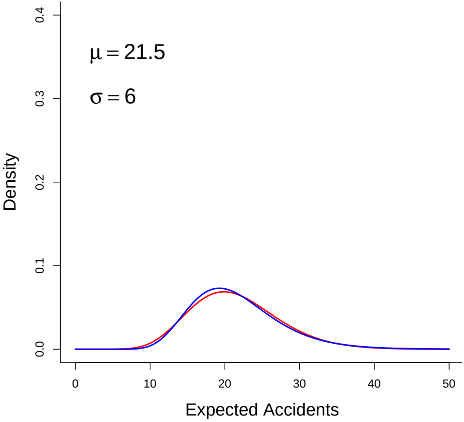
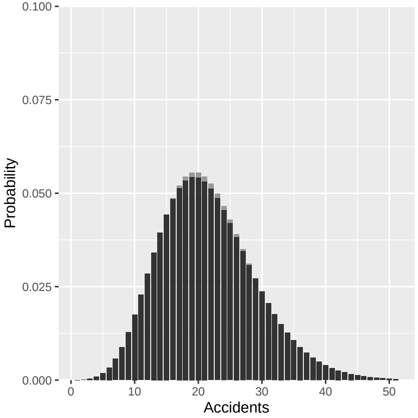




GLMM: Binomial Overdispersion

```
> photo_glm1 <- glm(cbind(g5, l5) ~ type + ypub, data = photo_long,
+     family = binomial)
```

GLMM: Binomial Overdispersion

```
> photo_glm1 <- glm(cbind(g5, l5) ~ type + ypub, data = photo_long,
+     family = binomial)
> summary(photo_glm1)
Deviance Residuals:
     Min        1Q    Median        3Q       Max
-11.3044   -3.1800   -0.7432    2.9375   12.0952

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -0.607223   0.068161  -8.909  < 2e-16 ***
typehappy   -1.173532   0.061646 -19.037  < 2e-16 ***
ypub         0.020165   0.002473   8.154 3.54e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 1547.7  on 43  degrees of freedom
Residual deviance: 1101.7  on 41  degrees of freedom
AIC: 1316.2

Number of Fisher Scoring iterations: 4
```

GLMM: Binomial Overdispersion

```
> photo_long$obs <- as.factor(1:nrow(photo_long))
```

GLMM: Binomial Overdispersion

```
> photo_long$obs <- as.factor(1:nrow(photo_long))

> photo_glm3 <- glmer(cbind(g5, l5) ~ type + ypub + (1 | obs),
+     data = photo_long, family = binomial)
```

GLMM: Binomial Overdispersion

```
> photo_long$obs <- as.factor(1:nrow(photo_long))

> photo_glm3 <- glmer(cbind(g5, l5) ~ type + ypub + (1 | obs),
+      data = photo_long, family = binomial)

> summary(photo_glm3)
     AIC       BIC    logLik deviance df.resid
   397.8     405.0    -194.9    389.8       40

Scaled residuals:
     Min       1Q   Median       3Q      Max
-0.80571 -0.16940 -0.00958  0.10938  0.67913

Random effects:
 Groups Name        Variance Std.Dev.
 obs    (Intercept) 1.177    1.085
Number of obs: 44, groups:  obs, 44

Fixed effects:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -0.66422    0.39058  -1.701 0.089022 .
typehappy   -1.27926    0.33519  -3.817 0.000135 ***
ypub         0.02019    0.01385   1.458 0.144824
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

GLMM: Binomial Overdispersion

```
> head(photo_long_full)

  y respondent photo  type  person age ypub
1 5          1  4510 happy peter_k  57   34
2 1          2  4510 happy peter_k  57   34
3 3          3  4510 happy peter_k  57   34
4 7          4  4510 happy peter_k  57   34
5 1          5  4510 happy peter_k  57   34
6 2          6  4510 happy peter_k  57   34
```

GLMM: Binomial Overdispersion

```
> head(photo_long_full)

  y respondent photo  type  person age ypub
1 5          1  4510 happy peter_k  57   34
2 1          2  4510 happy peter_k  57   34
3 3          3  4510 happy peter_k  57   34
4 7          4  4510 happy peter_k  57   34
5 1          5  4510 happy peter_k  57   34
6 2          6  4510 happy peter_k  57   34

> photo_long_full$g5 <- as.numeric(photo_long_full$y > 5)
```

GLMM: Binomial Overdispersion

```
> photo_glm4 <- glmer(g5 ~ type + ypub + (1 | photo), data = photo_long_full,
+     family = binomial)
```

GLMM: Binomial Overdispersion

```
> photo_glm4 <- glmer(g5 ~ type + ypub + (1 | photo), data = photo_long_full,
+     family = binomial)

> summary(photo_glm4)
     AIC      BIC   logLik deviance df.resid
  5480.7   5507.0  -2736.3   5472.7     5346

Scaled residuals:
    Min      1Q  Median      3Q     Max
-2.9925 -0.5432 -0.3660  0.6313  3.7175

Random effects:
 Groups Name        Variance Std.Dev.
 photo  (Intercept) 1.177    1.085
Number of obs: 5350, groups:  photo, 44

Fixed effects:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -0.66424    0.39049  -1.701 0.088930 .
typehappy   -1.27925    0.33530  -3.815 0.000136 ***
ypub         0.02019    0.01384   1.458 0.144739
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Why do we have (1|...)?

```
(1|nest_orig)
```

Why do we have (1|...)?

```
(1|nest_orig)
```

- Think of the left hand side as a model formula:

Why do we have (1|...)?

(1|nest_orig)

- Think of the left hand side as a model formula:

```
> head(model.matrix(~1, data = BTtarsus))
  (Intercept)
1           1
2           1
3           1
4           1
5           1
6           1
```

Why do we have (1|...)?

(1|nest_orig)

- Think of the left hand side as a model formula:

```
> head(model.matrix(~1, data = BTtarsus))
  (Intercept)
1           1
2           1
3           1
4           1
5           1
6           1
```

	11_A9	11_A7	11_A6	11_A46	11_A44	...
(In)	(In).11_A9	(In).11_A7	(In).11_A6	(In).11_A46	(In).11_A44	...

Structured Random Effects

```
(sex-1|nest_orig)
```

- Think of the left hand side as a model formula:

Structured Random Effects

```
(sex-1|nest_orig)
```

- Think of the left hand side as a model formula:

```
> head(model.matrix(~sex - 1, data = BTtarsus))
  sexF sexM
1    1    0
2    0    1
3    1    0
4    0    1
5    0    1
6    0    1
```

Structured Random Effects

`(sex-1|nest_orig)`

- Think of the left hand side as a model formula:

```
> head(model.matrix(~sex - 1, data = BTtarsus))
  sexF sexM
1    1    0
2    0    1
3    1    0
4    0    1
5    0    1
6    0    1
```

	11_A9	11_A7	11_A6	11_A46	11_A44	...
sexF	sexF.11_A9	sexF.11_A7	sexF.11_A6	sexF.11_A46	sexF.11_A44	...
sexM	sexM.11_A9	sexM.11_A7	sexM.11_A6	sexM.11_A46	sexM.11_A44	...

Structured Random Effects

`(sex-1|nest_orig)`

	11_A9	11_A7	11_A6	11_A46	11_A44	...
sexF	sexF.11_A9	sexF.11_A7	sexF.11_A6	sexF.11_A46	sexF.11_A44	...
sexM	sexM.11_A9	sexM.11_A7	sexM.11_A6	sexM.11_A46	sexM.11_A44	...

Structured Random Effects

(sex-1|nest_orig)

	11_A9	11_A7	11_A6	11_A46	11_A44	...
sexF	sexF.11_A9	sexF.11_A7	sexF.11_A6	sexF.11_A46	sexF.11_A44	...
sexM	sexM.11_A9	sexM.11_A7	sexM.11_A6	sexM.11_A46	sexM.11_A44	...

$$\mathbf{V}_{\texttt{nest\_orig}} = \begin{bmatrix} \sigma^2_{\texttt{Female}} & \sigma_{\texttt{Female,Male}} \\ \sigma_{\texttt{Female,Male}} & \sigma^2_{\texttt{Male}} \end{bmatrix}$$

Structured Random Effects

`(sex-1|nest_orig)`

	11_A9	11_A7	11_A6	11_A46	11_A44	...
sexF	sexF.11_A9	sexF.11_A7	sexF.11_A6	sexF.11_A46	sexF.11_A44	...
sexM	sexM.11_A9	sexM.11_A7	sexM.11_A6	sexM.11_A46	sexM.11_A44	...

$$\mathbf{V}_{\texttt{nest\_orig}} = \begin{bmatrix} \sigma^2_{\texttt{Female}} & \sigma_{\texttt{Female,Male}} \\ \sigma_{\texttt{Female,Male}} & \sigma^2_{\texttt{Male}} \end{bmatrix}$$

$$r_{\texttt{Female,Male}} = \frac{\sigma_{\texttt{Female,Male}}}{\sigma_{\texttt{Male}}\sigma_{\texttt{Female}}}$$

Structured Random Effects

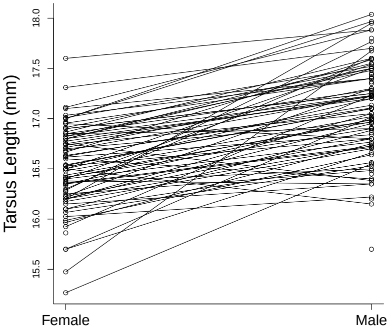


Figure: Average tarsus lengths for daughters and sons. The lines join sisters and their brothers.

Structured Random Effects

```
> tarsus_m6 <- lmer(tarsus_mm ~ sex + day_hatch + year +
+     (sex - 1 | nest_orig) + (sex - 1 | nest_rear),
+     data = BTtarsus)
```

Structured Random Effects

```
> tarsus_m6 <- lmer(tarsus_mm ~ sex + day_hatch + year +
+      (sex - 1 | nest_orig) + (sex - 1 | nest_rear),
+      data = BTtarsus)
```

```
> summary(tarsus_m6)
```

```
REML criterion at convergence: 3489.3

Scaled residuals:
    Min      1Q  Median      3Q     Max
-5.2061 -0.5752  0.0107  0.6199  3.2066

Random effects:
 Groups    Name Variance Std.Dev. Corr
 nest_orig sexF 0.09274  0.3045
           sexM 0.06976  0.2641   1.00
 nest_rear sexF 0.12129  0.3483
           sexM 0.14823  0.3850   1.00
 Residual       0.12901  0.3592
Number of obs: 2908, groups:  nest_orig, 440; nest_rear, 358
```

Structured Random Effects

```
> tarsus_m7 <- lmer(tarsus_mm ~ sex + day_hatch + year +
+     (1 | nest_orig) + (sex - 1 | nest_rear), data = BTtarsus)
```

Structured Random Effects

```
> tarsus_m7 <- lmer(tarsus_mm ~ sex + day_hatch + year +
+     (1 | nest_orig) + (sex - 1 | nest_rear), data = BTtarsus)

> anova(tarsus_m6, tarsus_m7)
          npar    AIC    BIC  logLik deviance  Chisq Df
tarsus_m7   11 3483.9 3549.6 -1731.0   3461.9
tarsus_m6   13 3485.0 3562.7 -1729.5   3459.0 2.8855  2
          Pr(>Chisq)
tarsus_m7
tarsus_m6     0.2363
```

Structured Random Effects

```
> tarsus_m7 <- lmer(tarsus_mm ~ sex + day_hatch + year +
+     (1 | nest_orig) + (sex - 1 | nest_rear), data = BTtarsus)

> anova(tarsus_m6, tarsus_m7)
          npar    AIC    BIC  logLik deviance  Chisq Df
tarsus_m7   11 3483.9 3549.6 -1731.0   3461.9
tarsus_m6   13 3485.0 3562.7 -1729.5   3459.0 2.8855  2
          Pr(>Chisq)
tarsus_m7
tarsus_m6     0.2363

> confint(tarsus_m6, ".sig02")
          2.5 % 97.5 %
.sig02 0.916107      1
```

Structured Random Effects: Random Regression

```
(1+day_hatch|nest_rear)
```

- Think of the left hand side as a model formula:

Structured Random Effects: Random Regression

```
(1+day_hatch|nest_rear)
```

- Think of the left hand side as a model formula:

```
> head(model.matrix(~1 + day_hatch, data = BTtarsus))
    (Intercept) day_hatch
100           1         1
7             1         3
200           1         0
300           1         0
400           1         1
500           1         0
```

Structured Random Effects: Random Regression

`(1+day_hatch|nest_rear)`

- Think of the left hand side as a model formula:

```
> head(model.matrix(~1 + day_hatch, data = BTtarsus))
    (Intercept) day_hatch
100           1         1
7             1         3
200           1         0
300           1         0
400           1         1
500           1         0
```

	11_A9	11_A7	11_A6	11_A46	11_A44	...
(In)	(In).11_A9	(In).11_A7	(In).11_A6	(In).11_A46	(In).11_A44	...
dayh	dayh.11_A9	dayh.11_A7	dayh.11_A6	dayh.11_A46	dayh.11_A44	...

Structured Random Effects: Random Regression

(1+day_hatch|nest_rear)

	11_A9	11_A7	11_A6	11_A46	11_A44	...
(In)	(In).11_A9	(In).11_A7	(In).11_A6	(In).11_A46	(In).11_A44	...
dayh	dayh.11_A9	dayh.11_A7	dayh.11_A6	dayh.11_A46	dayh.11_A44	...

Structured Random Effects: Random Regression

(1+day_hatch|nest_rear)

	11_A9	11_A7	11_A6	11_A46	11_A44	...
(In)	(In).11_A9	(In).11_A7	(In).11_A6	(In).11_A46	(In).11_A44	...
dayh	dayh.11_A9	dayh.11_A7	dayh.11_A6	dayh.11_A46	dayh.11_A44	...

$$\mathbf{V}_{\texttt{nest\_rear}} = \begin{bmatrix} \sigma^2_{\texttt{(In)}} & \sigma_{\texttt{(In),dayh}} \\ \sigma_{\texttt{(In),dayh}} & \sigma^2_{\texttt{dayh}} \end{bmatrix}$$

Structured Random Effects: Random Regression

(1+day_hatch|nest_rear)

	11_A9	11_A7	11_A6	11_A46	11_A44	...
(In)	(In).11_A9	(In).11_A7	(In).11_A6	(In).11_A46	(In).11_A44	...
dayh	dayh.11_A9	dayh.11_A7	dayh.11_A6	dayh.11_A46	dayh.11_A44	...

$$\mathbf{V}_{\texttt{nest\_rear}} = \begin{bmatrix} \sigma^2_{\texttt{(In)}} & \sigma_{\texttt{(In),dayh}} \\ \sigma_{\texttt{(In),dayh}} & \sigma^2_{\texttt{dayh}} \end{bmatrix}$$

$$r_{\texttt{(In),dayh}} = \frac{\sigma_{\texttt{(In),dayh}}}{\sigma_{\texttt{(In)}}\sigma_{\texttt{dayh}}}$$

Structured Random Effects: Random Regression

```
> tarsus_m8 <- lmer(tarsus_mm ~ sex + day_hatch +
+     year + (1 | nest_orig) + (1 + day_hatch |
+     nest_rear), data = BTtarsus)
```

Structured Random Effects: Random Regression

```
> tarsus_m8 <- lmer(tarsus_mm ~ sex + day_hatch +
+     year + (1 | nest_orig) + (1 + day_hatch |
+     nest_rear), data = BTtarsus)
> summary(tarsus_m8)

REML criterion at convergence: 3440.2

Scaled residuals:
    Min      1Q  Median      3Q     Max
-5.2571 -0.5655  0.0145  0.6023  3.3925

Random effects:
 Groups    Name        Variance Std.Dev. Corr
 nest_orig (Intercept) 0.08006  0.2830
 nest_rear (Intercept) 0.13064  0.3614
           day_hatch   0.02316  0.1522   0.15
 Residual              0.12030  0.3468
Number of obs: 2908, groups:
nest_orig, 440; nest_rear, 358

Fixed effects:
```

Structured Random Effects: Random Regression

```
> anova(tarsus_m8, tarsus_m5)
          npar    AIC    BIC  logLik deviance Chisq Df
tarsus_m5    9 3481.4 3535.2 -1731.7   3463.4
tarsus_m8   11 3432.6 3498.3 -1705.3   3410.6 52.84  2
          Pr(>Chisq)
tarsus_m5
tarsus_m8  3.356e-12 ***
---
Signif. codes:
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

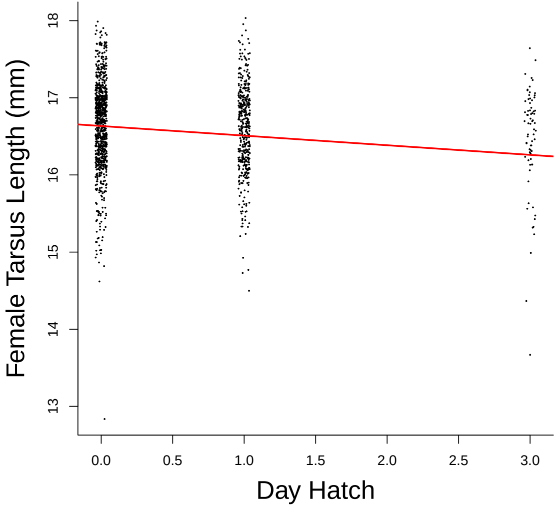
Structured Random Effects: Random Regression

```
> anova(tarsus_m8, tarsus_m5)
          npar    AIC    BIC  logLik deviance Chisq Df
tarsus_m5    9 3481.4 3535.2 -1731.7   3463.4
tarsus_m8   11 3432.6 3498.3 -1705.3   3410.6 52.84  2
          Pr(>Chisq)
tarsus_m5
tarsus_m8  3.356e-12 ***
---
Signif. codes:
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

> confint(tarsus_m8, ".sig04")^2
             2.5 %      97.5 %
.sig04 0.01344515 0.03572781
```

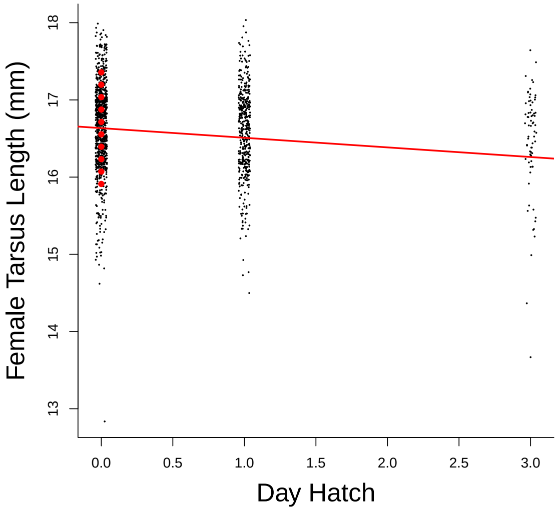
Structured Random Effects: Random Regression

```
> anova(tarsus_m8, tarsus_m5)
          npar    AIC    BIC  logLik deviance Chisq Df
tarsus_m5    9 3481.4 3535.2 -1731.7   3463.4
tarsus_m8   11 3432.6 3498.3 -1705.3   3410.6 52.84  2
          Pr(>Chisq)
tarsus_m5
tarsus_m8  3.356e-12 ***
---
Signif. codes:
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

> confint(tarsus_m8, ".sig04")^2
            2.5 %     97.5 %
.sig04 0.01344515 0.03572781
> confint(tarsus_m8, ".sig03")
            2.5 %    97.5 %
.sig03 -0.1619683 0.4865619
```

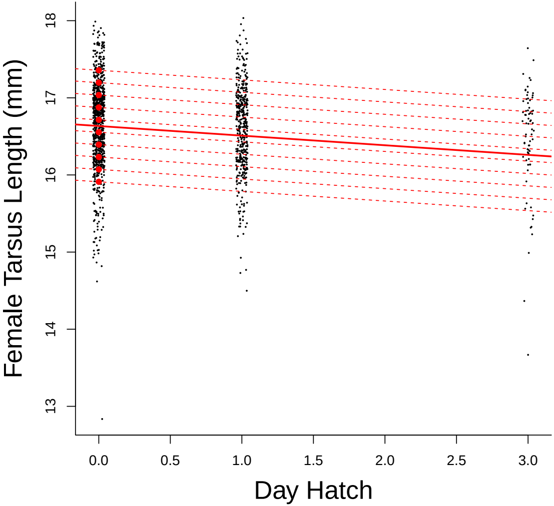
Structured Random Effects: Random Regression



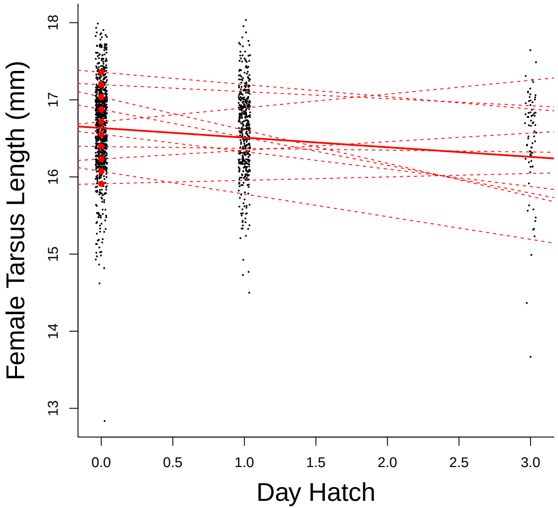
Structured Random Effects: Random Regression



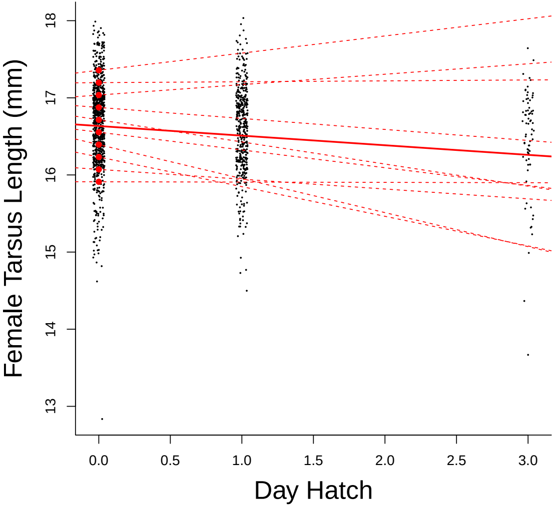
Structured Random Effects: Random Regression



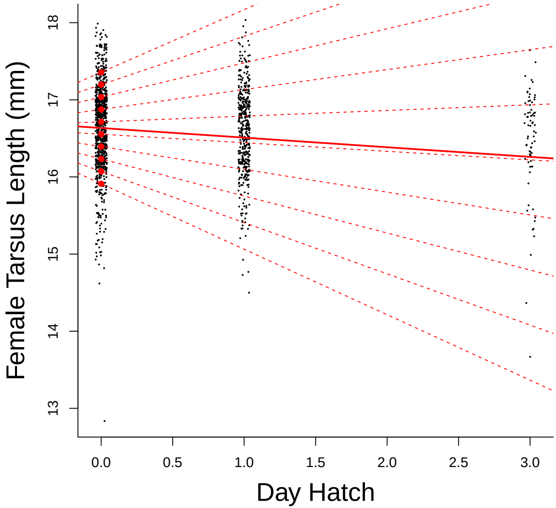
Structured Random Effects: Random Regression



Structured Random Effects: Random Regression



Structured Random Effects: Random Regression



Other Covariance Structures

- autoregressive (time-series analysis)
- exponential (spatial analysis)
- pedigree (animal model)
- phylogeny (comparative analysis)
- measurement error (meta-analysis)
- multi-membership models
- multi-response models